LIE SYMMETRY ANALYSIS AND CONSERVATION LAWS OF SYSTEMS OF ELLIPTIC EQUATIONS

MOSITO LEKHOOANA

LIE SYMMETRY ANALYSIS AND CONSERVATION LAWS OF SYSTEMS OF ELLIPTIC EQUATIONS

MOSITO LEKHOOANA

A dissertation submitted to the Faculty of Science and Technology, National University of Lesotho, in partial fulfillment of the requirements for the degree of Master of Science.

March 2021

DECLARATION

I, Mosito Lekhooana, student number 200900836, declare that this dissertation submitted for the degree of Master of Science in Applied Mathematics at National University of Lesotho has not previously been submitted by me for a degree at this or any other University. Further, I declare that this is my original work and any work done by others has been acknowledged accordingly.

____

M. Lekhooana

_____ day of _______________ 20 _____

ABSTRACT

This work involves the study of elliptic type systems of equations in three independent variables. In the first part of the work, the Lie point symmetries of the systems are obtained; some of the symmetries of a system are used to perform reduction to an invariant system with one less independent variable. The symmetries of the invariant system are also obtained and used for further reduction to a system with one more less independent variable. The invariant solutions of the last reduced system are constructed. The second part of this study deals with the construction of conservation laws of the systems of elliptic equations. The variational approach is used, that is, the Noether point symmetries and their corresponding conserved vectors are obtained.

DEDICATION

To my family.

ACKNOWLEDGEMENTS

I would like to convey special thanks to Associate Professor M. Molati for his guidance throughout this work, without your support this work would not have been completed. Secondly to all my colleagues whose workloads had to be increased as I continued my studies, your unconditional sacrifices are greatly appreciated. I would also like to appreciate financial support of the tuition waiver from the National University of Lesotho. Lastly to my whole family, your day to day presence by me gave me strength.

Contents

List of Tables

Chapter 1

Introduction

Differential equations (DEs) are widely used as models to describe different situations in science and engineering. A mathematical model can be represented by a single partial differential equation (PDE) or ordinary differential equation (ODE); many situations describing real life problems are modelled by systems of PDEs or ODEs or both. A second-order partial differential equation of the form

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+D(x,y)u_x+E(x,y)u_y+F(x,y)u=G(x,y), \tag{1.1}$$

in two independent variables x and y and one dependent variable u is elliptic if $AC-B^2>0$. Elliptic equations [9, 18, 24] are a class of PDEs that describe a phenomena that do not change from moment to moment, e.g., heat or fluid flow within a medium with no accumulations. Mathematics areas including harmonic analysis apply the concept of elliptic equations [7]. A classical prototype of a linear elliptic equation dates back to a well-known French mathematician, Pierre-Simon Laplace (1749-1827) and his famous Laplace equation, and its non-homogeneous counterpart, Poisson's equation [9, 10]. The Laplace equation is obtained from Eq. (1.1) by setting $B=D=E=F=G=0$ and setting $A=C=1$. In general, it is difficult to solve Eq. (1.1) but an appropriate transformation from (x,y) to (λ,μ) reduces Eq. (1.1) to the canonical form

$$u_{\lambda\lambda}+u_{\mu\mu}=H(\lambda,\mu,u,u_\lambda,u_\mu). \tag{1.2}$$

At times several processes take place at the same time and they somehow interact. On the other hand, one process may require a response (or input) from another in order to kick start. Both phenomena can be modelled by a system of coupled DEs. Depending on how the processes interact, the system may either be weakly coupled or strongly coupled [16, 17, 26].

1.1 Systems of elliptic equations

Systems of elliptic equations investigated in this dissertation are derived from the following concepts;

1. **Real Jordan Canonical forms for** 2×2 **matrices**

   Definition 1.1.1 Two square matrices A and B are similar if there is an invertible matrix P such that $B=PAP^{-1}$.

   Definition 1.1.2 The characteristic polynomial of a matrix A is the polynomial $\det(\lambda I-A)$.

Theorem 1.1.3 An $n \times n$ matrix A is diagonalizable if its minimum polynomial has only linear factors.

Theorem 1.1.4 Let $A \in M_2(\mathbb{R})$, then there exits a matrix $B \in M_2(\mathbb{R})$ such that $B = PAP^{-1}$ where $P \in M_2(\mathbb{R})$ and B assumes one of the following cases :

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \lambda_1 \neq \lambda_2, \tag{1.3a}$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{1.3b}$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{1.3c}$$

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}; \alpha^2 + \beta^2 \neq 0 \text{ and } \alpha, \beta \neq 0, \tag{1.3d}$$

where $\alpha, \beta, \lambda, \lambda_1, \lambda_2$ are real numbers. Case (1.3a) corresponds to a situation where a 2×2 matrix A has distinct real eigenvalues λ_1 and λ_2. This guarantees that A is diagonalizable, or equivalently we can say A is similar to some matrix B, i.e. $PAP^{-1} = B$ where B is a diagonal matrix. For eigenvalues λ_1 and λ_2, there correspond eigenvectors $\underline{v}_1$ and $\underline{v}_2$ respectively satisfying the eigenvector equations

$$(A - \lambda_i I)\underline{v}_i = \underline{0}, \text{ where } i = 1, 2. \tag{1.4}$$

Cases (1.3b) and (1.3c) are those in which A has repeated real eigenvalues (i.e., λ is a of multiplicity 2). These two cases are distinguished by the dimension of the respective null space, case (1.3b) has dimension 2 and case (1.3c) it is 1. The corresponding eigenvectors satisfy

$$(A - \lambda I)\underline{v}_1 = 0, \tag{1.5a}$$

$$(A - \lambda I)\underline{v}_2 = \underline{v}_1 \text{ or } (A - \lambda I)^2 \underline{v}_2 = 0. \tag{1.5b}$$

The last case happens when A has complex conjugate eigenvalues $\lambda = \alpha \pm i\beta$ of the coefficient matrix A with corresponding eigenvectors $\underline{v} = \underline{a} \pm i\underline{b}$.

2. **Canonical forms of systems of DEs**

Consider the second-order system of DEs in the matrix form $\Delta \begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$, where $A \in M_2(\mathbb{R})$ and Δ is the n-dimensional Laplacian ($n \geq 2$).

Let $\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix}$, where P is the matrix introduced in theorem 1.1.4. Elementary computations lead to $\Delta \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = B \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix}$, where B is the placeholder for one of the matrices in (1.3). Appropriate spacial scalings leads to the following canonical forms

$$\begin{aligned} \Delta u &= u, \\ \Delta v &= \alpha v; \quad \alpha \neq 1. \end{aligned} \tag{1.6a}$$

$$\Delta u = -u, \quad \Delta v = \alpha v; \quad \alpha \neq -1. \tag{1.6b}$$

$$\Delta u = 0, \quad \Delta v = 0. \tag{1.6c}$$

$$\Delta u = u, \quad \Delta v = v. \tag{1.6d}$$

$$\Delta u = -u, \quad \Delta v = -v. \tag{1.6e}$$

$$\Delta u = \alpha u + v, \quad \Delta v = v; \ \alpha > 0. \tag{1.6f}$$

$$\Delta u = -\alpha u - v, \quad \Delta v = -v; \ \alpha > 0. \tag{1.6g}$$

$$\Delta u = \alpha u + \beta v, \quad \Delta v = -\beta u + \alpha v; \quad \alpha^2 + \beta^2 = 1. \tag{1.6h}$$

In the next Chapter we study the Lie symmetries [3, 22, 23] of the above systems with the help of YaLie package [5]. Lie symmetries of each system are obtained for $n = 3$ (i.e., we consider the 3-dimensional Laplacian) but first we present the preliminaries of the Lie symmetry method.

1.2 Lie symmetry method

1.2.1 Preliminaries

Here-in we present useful information adopted from [19].

Definition 1.2.1 A k^{th}-order ($k \geq 1$) system E of s DEs is defined by

$$E^\sigma(x, u, u_{(1)}, ..., u_{(k)}) = 0, \ \sigma = 1, ..., s, \tag{1.7}$$

where $u \equiv (u^1, u^2, ..., u^q)$ is the dependent vector, $x \equiv (x^1, x^2, ..., x^n)$ is the independent vector and $u_{(1)}, u_{(2)}, ..., u_{(k)}$ are respectively the collection of all first, second, up to k^{th}-order derivatives.

Definition 1.2.2 A symmetry transformation of the system (1.7) is an invertible transformation of the variables x and u, namely

$$\bar{x}^i = f^i(x,u),\ \bar{u}^\alpha = \phi^\alpha(x,u),\ i = 1,...,n;\ \alpha = 1,...,q,$$

that leaves (1.7) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^\alpha(\bar{x},\bar{u},\bar{u}_{(1)},...,\bar{u}_{(k)}) = 0,\ \sigma = 1,...,s,$$

whenever (1.7) is satisfied.

Definition 1.2.3 A set G of transformations

$$T_a : \bar{x}^i = f^i(x,u,a), \bar{u}^\alpha = \phi^\alpha(x,u,a), i = 1,...,n; \alpha = 1,...,q, \tag{1.8}$$

is called a continuous one-parameter (local) Lie-group of transformations in $\mathbb{R}^{n+q}$ provided the group properties of closure, identity and inverses are satisfied. Here f^i and ϕ^α are differentiable functions and a is a real parameter which continuously takes values in a neighbourhood $\mathbb{D} \subseteq \mathbb{R}$ of $a = 0$.

Definition 1.2.4 An infinitesimal generator X of the group transformations G (1.8) is the differential operator of the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{1.9}$$

such that

$$\bar{x}^i = x^i + a\xi^i(x,u) + O(a^2) = (1 + aX)x^i,\ \bar{u}^\alpha = u^\alpha + a\eta^\alpha(x,u) + O(a^2) = (1 + aX)u^\alpha. \tag{1.10}$$

Here and throughout this section, the Einstein summation convention is adopted. The one-parameter group elements (1.10) are known as the *infinetesimal transformations* obtained from (1.8) by first-order (Taylor expansion) approximations around parameter $a = 0$. The operator (1.9) is also called the vector field or the Lie point symmetry generator (or operator).

Definition 1.2.5 The extended infinitesimal generator $X^{[k]}$ of the k^{th} prolonged (extended) group $G^{[k]}$ on the space $(x,u,...,u_{(k)})$ is called the k^{th} prolongation of X, viz,

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta^\alpha_i(x,u,u_{(1)})\frac{\partial}{\partial u^\alpha_i} + ... + \zeta^\alpha_{(i_1...i_k)}\frac{\partial}{\partial u^\alpha_{(i_1...i_k)}}.$$

The coefficients $\zeta^\alpha s$ are defined recursively by the prolongation formulae

$$\begin{aligned}
\zeta^\alpha_i &= D_i(\eta^\alpha) - u^\alpha_{(j)} D_i(\xi^j),\\
\zeta^\alpha_{ij} &= D_j(\zeta^\alpha_i) - u^\alpha_{(il)} D_j(\xi^l),\\
&\ \,.\\
&\ \,.\\
&\ \,.\\
\zeta^\alpha_{i_1...i_k} &= D_{i_k}(\zeta^\alpha_{i_1...i_{k-1}}) - u^\alpha_{(i_1...i_{k-1}l)} D_{i_k}(\xi^l),
\end{aligned}$$

where

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_{(j)}\frac{\partial}{\partial u^\alpha} + u^\alpha_{(jk)}\frac{\partial}{\partial u^\alpha_{(k)}}...;\ u^\alpha_{(j)} = D_j(u^\alpha),\ u^\alpha_{(jk)} = D_j(u^\alpha_{(k)}),$$

is the total derivative operator with respect to x^i.

Theorem 1.2.6 Let G be a group of infinitesimal transformations (1.10) admitted by a system E. Then G consists of symmetries of the system E if and only if

$$X^{[k]}\Big(E^{\sigma}(x,u,u_{(1)},...,u_{(k)})\Big)=0,\ \sigma=1,...,s, \tag{1.11}$$

whenever (1.7) is satisfied for every group generator X of G. The symmetry conditions (1.11), also called the invariance criterion, can be written compactly as

$$X^{[k]}\Big(E^{\sigma}(x,u,u_{(1)},...,u_{(k)})\Big)\Big|_{(1.7)}=0,\ \sigma=1,...,s, \tag{1.12}$$

where $\big|_{(1.7)}$ means evaluated on the surface (1.7). Eq. (1.12) are the so-called determining equations. This theorem summarizes the Lie's algorithm.

In general the determining equations comprise an over-determined system of linear homogeneous PDEs for the unknown coordinated ξ^i and η^α of the symmetry generator X. The solutions of the determining system form a vector space, that is, any finite linear combination of symmetries is again a symmetry. This stems from the fact that the determining equations are linear. Looking at the DE one can easily deduce the trivial symmetries such as translations, the other symmetries can be computed by using the Lie's algorithm.

1.2.2 Calculation of symmetries

The Lie's algorithm customized to the current work is presented below. Consider the second-order system E of PDEs

$$E(x,y,z,u_x,v_x,u_y,v_y,u_z,v_z,u_{xx},v_{xx},u_{yy},v_{yy},u_{zz},v_{zz},u_{xy},v_{xy},u_{xz},v_{xz},u_{yz},v_{yz})=0, \tag{1.13}$$

where x,y,z are independent variables and u,v are dependent variables. The operator

$$\begin{aligned}X=&\xi^1(x,y,z,u,v)\frac{\partial}{\partial_x}+\xi^2(x,y,z,u,v)\frac{\partial}{\partial_y}+\xi^3(x,y,z,u,v)\frac{\partial}{\partial_z}+\eta^1(x,y,z,u,v)\frac{\partial}{\partial_u}\\&+\eta^2(x,y,z,u,v)\frac{\partial}{\partial_v},\end{aligned} \tag{1.14}$$

is the symmetry generator of (1.13) provided

$$X^{[2]}\Big(E\Big)\Big|_{(1.13)}=0. \tag{1.15}$$

The operator $X^{[2]}$ is the second prolongation of X defined by

$$\begin{aligned}X^{[2]}=X^{[1]}&+\zeta^1_{xx}\frac{\partial}{\partial_{u_{xx}}}+\zeta^1_{xy}\frac{\partial}{\partial_{u_{xy}}}+\zeta^1_{xz}\frac{\partial}{\partial_{u_{xz}}}+\zeta^1_{yy}\frac{\partial}{\partial_{u_{yy}}}+\zeta^1_{yz}\frac{\partial}{\partial_{u_{yz}}}\\&+\zeta^1_{zz}\frac{\partial}{\partial_{u_{zz}}}+\zeta^2_{xx}\frac{\partial}{\partial_{u_{xx}}}+\zeta^2_{xy}\frac{\partial}{\partial_{u_{xy}}}+\zeta^2_{xz}\frac{\partial}{\partial_{u_{xz}}}+\zeta^2_{yy}\frac{\partial}{\partial_{u_{yy}}}+\zeta^2_{yz}\frac{\partial}{\partial_{u_{yz}}}+\zeta^2_{zz}\frac{\partial}{\partial_{u_{zz}}},\end{aligned} \tag{1.16}$$

where

$$
\begin{aligned}
\zeta^1_{xx} = {} & v_{xx}\eta^1_v + u_{xx}\eta^1_u - 2u_{xy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - 2u_{xz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) \\
& + v_x\eta^1_{xv} - u_x\left(v_{xx}\xi^1_v + u_{xx}\xi^1_u + v_x\xi^1_{xv} + v_x\left(v_x\xi^1_{vv} + u_x\xi^1_{uv} + \xi^1_{xv}\right) + \xi^1_{xx}\right) \\
& - u_y\left(v_{xx}\xi^2_v + u_{xx}\xi^2_u + v_x\xi^2_{xv} + v_x\left(v_x\xi^2_{vv} + u_x\xi^2_{uv} + \xi^2_{xv}\right) + u_x\xi^2_{xu} + \xi^2_{xx}\right) \\
& - u_z\left(v_{xx}\xi^3_v + u_{xx}\xi^3_u + v_x\xi^3_{xv} + v_x\left(v_x\xi^3_{vv} + u_x\xi^3_{uv} + \xi^3_{xv}\right) + u_x\xi^3_{xu} + \xi^3_{xx}\right) \\
& + v_x\left(v_x\eta^1_{vv} + u_x\eta^1_{uv} + \eta^1_{xv}\right) + u_x\eta^1_{xu} + u_x\left(v_x\eta^1_{uv} + u_x\eta^1_{uu} + \eta^1_{xu}\right) + \eta^1_{xx} + u_x\xi^1_{xu} \\
& - 2u_{xx}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - u_x^2\left(v_x\xi^1_{uv} + u_x\xi^1_{uu} + \xi^1_{xu}\right) - u_y u_x\left(v_x\xi^2_{uv} + u_x\xi^2_{uu} + \xi^2_{xu}\right) \\
& - u_z u_x\left(v_x\xi^3_{uv} + u_x\xi^3_{uu} + \xi^3_{xu}\right),
\end{aligned}
\tag{1.17a}
$$

$$
\begin{aligned}
\zeta^1_{xy} = {} & v_{xy}\eta^1_v + u_{xy}\eta^1_u - u_{xx}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - u_{xy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) \\
& - u_{yy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - u_{yz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) + v_y\eta^1_{xv} + \eta^1_{xy} + u_y\eta^1_{xu} \\
& - u_x\left(v_{xy}\xi^1_v + u_{xy}\xi^1_u + u_x\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) + v_y\xi^1_{xv} + u_y\xi^1_{xu} + \xi^1_{xy}\right) \\
& - u_y\left(v_{xy}\xi^2_v + u_{xy}\xi^2_u + u_x\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) + v_y\xi^2_{xv} + u_y\xi^2_{xu} + \xi^2_{xy}\right) \\
& - u_z\left(v_{xy}\xi^3_v + u_{xy}\xi^3_u + u_x\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + v_y\xi^3_{xv} + u_y\xi^3_{xu} + \xi^3_{xy}\right) \\
& + v_x\left(v_y\eta^1_{vv} + u_y\eta^1_{uv} + \eta^1_{yv}\right) + u_x\left(v_y\eta^1_{uv} + u_y\eta^1_{uu} + \eta^1_{yu}\right) - u_{xz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) \\
& - u_{xy}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - u_x v_x\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) - u_y v_x\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) \\
& - u_z v_x\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right),
\end{aligned}
\tag{1.17b}
$$

$$
\begin{aligned}
\zeta^1_{xz} = {} & v_{xz}\xi^1_v + u_{xz}\eta^1_u - u_{xy}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - u_{xz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) \\
& + u_x\left(v_z\eta^1_{uv} + u_z\eta^1_{uu} + \eta^1_{zu}\right) - u_{xz}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) + v_z\eta^1_{xv} + u_z\eta^1_{xu} + \eta^1_{xz} \\
& - u_x\left(v_{xz}\xi^1_v + u_{xz}\xi^1_u + u_x\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + v_z\xi^1_{xv} + u_z\xi^1_{xu} + \xi^1_{xz}\right) \\
& - u_y\left(v_{xz}\xi^2_v + u_{xz}\xi^2_u + u_x\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) + v_z\xi^2_{xv} + u_z\xi^2_{xu} + \xi^2_{xz}\right) \\
& - u_z\left(v_{xz}\xi^3_v + u_{xz}\xi^3_u + u_x\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + v_z\xi^3_{xv} + u_z\xi^3_{xu} + \xi^3_{xz}\right) \\
& - u_{yz}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - u_{zz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) - u_{xx}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) \\
& + v_x\left(v_z\eta^1_{vv} + u_z\eta^1_{uv} + \eta^1_{zv}\right) - u_x v_x\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) - u_y v_x\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) \\
& - u_z v_x\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right),
\end{aligned}
\tag{1.17c}
$$

$$
\begin{aligned}
\zeta^1_{yy} = {} & v_{yy}\xi^1_v + u_{yy}\eta^1_u - 2u_{yy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) - 2u_{yz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) + v_y\eta^1_{yv} \\
& + \eta^1_{yy} + u_y\eta^1_{yu} + u_y\left(v_y\eta^1_{uv} + u_y\eta^1_{uu} + \eta^1_{yu}\right) - u_x\left(v_{yy}\xi^1_v + u_{yy}\xi^1_u + v_y\xi^1_{yv}\right. \\
& \left. + u_y\xi^1_{yu} + u_y\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) + \xi^1_{yy}\right) - u_y\left(v_{yy}\xi^2_v + u_{yy}\xi^2_u + v_y\xi^2_{yv}\right. \\
& \left. + u_y\xi^2_{yu} + u_y\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) + \xi^2_{yy}\right) - u_z\left(v_{yy}\xi^3_v + u_{yy}\xi^3_u + v_y\xi^3_{yv}\right. \\
& \left. + v_y\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right) + u_y\xi^3_{yu} + u_y\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + \xi^3_{yy}\right) \\
& + v_y\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) + v_y\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) + v_y\left(v_y\eta^1_{vv} + u_y\eta^1_{uv} + \eta^1_{yv}\right) \\
& - 2u_{xy}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right),
\end{aligned}
\tag{1.17d}
$$

$$
\begin{aligned}
\zeta^1_{yz} = {} & v_{yz}\eta^1_v + u_{yz}\eta^1_u - u_{yy}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - u_{yz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^1_{yv} \\
& + u_y\left(v_z\eta^1_{uv} + u_z\eta^1_{uu} + \eta^1_{zu}\right) - u_{xz}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - u_{yz}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) \\
& + \eta^1_{yz} - u_x\left(v_{yz}\xi^1_v + u_{yz}\xi^1_u + u_y\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + v_z\xi^1_{yv} + u_z\xi^1_{yu} + \xi^1_{yz}\right) \\
& - u_y\left(v_{yz}\xi^2_v + u_{yz}\xi^2_u + u_y\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) + v_z\xi^2_{yv} + u_z\xi^2_{yu} + \xi^2_{yz}\right) \\
& - u_z\left(v_y\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + u_y\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + v_z\xi^3_{yv} + u_z\xi^3_{yu} + \xi^3_{yz}\right) \\
& + u_z\eta^1_{yu} + v_y\left(v_z\eta^1_{vv} + u_z\eta^1_{uv} + \eta^1_{zv}\right) - u_{xy}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) - u_{zz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) \\
& - u_xv_y\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) - u_yv_y\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) - u_z\left(v_{yz}\xi^3_v + u_{yz}\xi^3_u\right),
\end{aligned}
\tag{1.17e}
$$

$$
\begin{aligned}
\zeta^1_{zz} = {} & v_{zz}\eta^1_v + u_{zz}\eta^1_u - 2u_{yz}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - 2u_{zz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^1_{zv} \\
& + u_z\eta^1_{zu} + u_z\left(v_z\eta^1_{uv} + u_z\eta^1_{uu} + \eta^1_{zu}\right) + \eta^1_{zz} - u_x\left(v_{zz}\xi^1_v + u_{zz}\xi^1_u + v_z\xi^1_{zv}\right. \\
& \left. + u_z\xi^1_{zu} + u_z\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + \xi^1_{zz}\right) - u_y\left(v_{zz}\xi^2_v + u_{zz}\xi^2_u + v_z\xi^2_{zv}\right. \\
& \left. + u_z\xi^2_{zu} + u_z\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) + \xi^2_{zz}\right) - u_z\left(v_{zz}\xi^3_v + u_{zz}\xi^3_u + v_z\xi^3_{zv}\right. \\
& \left. + v_z\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + u_z\xi^3_{zu} + u_z\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + \xi^3_{zz}\right) \\
& - 2u_{xz}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) + v_z\left(v_z\eta^1_{vv} + u_z\eta^1_{uv} + \eta^1_{zv}\right) + v_z\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) \\
& + v_z\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right),
\end{aligned}
\tag{1.17f}
$$

$$
\begin{aligned}
\zeta^2_{xx} = {} & v_{xx}\eta^2_v + u_{xx}\eta^2_u - 2v_{xx}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - 2v_{xy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) + v_x\eta^2_{xv} \\
& + u_x\eta^2_{xu} + \eta^2_{xx} - v_x\left(v_{xx}\xi^1_v + u_{xx}\xi^1_u + v_x\xi^1_{xv} + v_x\left(v_x\xi^1_{vv} + u_x\xi^1_{uv} + \xi^1_{xv}\right) + u_x\xi^1_{xu}\right. \\
& \left. + u_x\left(v_x\xi^1_{uv} + u_x\xi^1_{uu} + \xi^1_{xu}\right) + \xi^1_{xx}\right) - v_y\left(v_{xx}\xi^2_v + u_{xx}\xi^2_u + v_x\xi^2_{xv} + u_x\xi^2_{xu}\right. \\
& \left. - v_z\left(v_{xx}\xi^3_v + u_{xx}\xi^3_u + v_x\xi^3_{xv} + u_x\xi^3_{xu} + u_x\left(v_x\xi^3_{uv} + u_x\xi^3_{uu} + \xi^3_{xu}\right) + \xi^3_{xx}\right)\right. \\
& + v_x\left(v_x\eta^2_{vv} + u_x\eta^2_{uv} + \eta^2_{xv}\right) + u_x\left(v_x\xi^2_{uv} + u_x\xi^2_{uu} + \xi^2_{xu}\right) + \xi^2_{xx}\big) - 2v_{xz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) \\
& + u_x\left(v_x\eta^2_{uv} + u_x\eta^2_{uu} + \eta^2_{xu}\right) - v_xv_y\left(v_x\xi^2_{vv} + u_x\xi^2_{uv} + \xi^2_{xv}\right) \\
& - v_xv_z\left(v_x\xi^3_{vv} + u_x\xi^3_{uv} + \xi^3_{xv}\right),
\end{aligned}
\tag{1.17g}
$$

$$
\begin{aligned}
\zeta^2_{xy} = {} & v_{xy}\eta^2_v + u_{xy}\eta^2_u - v_{xx}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - v_{xy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) \\
& + u_x\left(v_y\eta^2_{uv} + u_y\eta^2_{uu} + \eta^2_{yu}\right) - v_{xy}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - v_{yy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) \\
& - v_x\left(v_{xy}\xi^1_v + u_{xy}\xi^1_u + v_x\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) + v_y\xi^1_{xv} + u_y\xi^1_{xu} + \xi^1_{xy}\right) \\
& - v_y\left(v_{xy}\xi^2_v + u_{xy}\xi^2_u + v_x\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) + v_y\xi^2_{xv} + u_y\xi^2_{xu} + \xi^2_{xy}\right) \\
& - v_z\left(v_{xy}\xi^3_v + u_{xy}\xi^3_u + v_x\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right) + v_y\xi^3_{xv} + u_y\xi^3_{xu} + \xi^3_{xy}\right) \\
& - v_{xz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) + v_x\left(v_y\eta^2_{vv} + u_y\eta^2_{uv} + \eta^2_{yv}\right) - v_{yz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) \\
& - u_xv_x\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) - u_xv_y\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) \\
& - u_xv_z\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + v_y\eta^2_{xv} + u_y\eta^2_{xu} + \eta^2_{xy},
\end{aligned}
\tag{1.17h}
$$

$$
\begin{aligned}
\zeta^2_{xz} =& v_{xz}\eta^2_v + u_{xz}\eta^2_u - v_{xx}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) - v_{xz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^2_{xv} \\
&+ u_x\left(v_z\eta^2_{uv} + u_z\eta^2_{uu} + \eta^2_{zu}\right) - v_{xz}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - v_{yz}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) \\
&+ \eta^2_{xz} - v_x\left(v_{xz}\xi^1_v + u_{xz}\xi^1_u + u_x\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + \ v_z\xi^1_{xv} + u_z\xi^1_{xu} + \xi^1_{xz}\right) \\
&- v_y\left(v_{xz}\xi^2_v + u_{xz}\xi^2_u + v_x\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) + \ v_z\xi^2_{xv} + u_z\xi^2_{xu} + \xi^2_{xz}\right) \\
&- v_z\left(v_{xz}\xi^3_v + u_{xz}\xi^3_u + v_x\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + \ v_z\xi^3_{xv} + u_z\xi^3_{xu} + \xi^3_{xz}\right) \\
&- v_{xy}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) + u_z\eta^2_{xu} + v_x\left(v_z\eta^2_{vv} + u_z\eta^2_{uv} + \eta^2_{zv}\right) - v_{zz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) \\
&- v_x^2\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) - u_x v_y\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) - u_x v_z\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right),
\end{aligned}
\tag{1.17i}
$$

$$
\begin{aligned}
\zeta^2_{yy} =& v_{yy}\eta^2_v + u_{yy}\eta^2_u - 2v_{xy}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - 2v_{yy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) \\
&- v_x\left(v_y\xi^1_{yv} + v_y\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) + u_y\xi^1_{yu} + \ u_y\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) + \xi^1_{yy}\right) \\
&- v_y\left(v_y\xi^2_{yv} + v_y\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) + u_y\xi^2_{yu} + \ u_y\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) + \xi^2_{yy}\right) \\
&- v_z\left(v_y\xi^3_{yv} + v_y\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right) + u_y\xi^3_{yu} + \ u_y\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + \xi^3_{yy}\right) \\
&+ v_y\left(v_y\eta^2_{vv} + u_y\eta^2_{uv} + \eta^2_{yv}\right) + u_y\eta^2_{yu} + u_y\left(v_y\eta^2_{uv} + u_y\eta^2_{uu} + \eta^2_{yu}\right) + \eta^2_{yy} + v_y\eta^2_{yv} \\
&- v_x(v_{yy}\xi^1_v + u_{yy}\xi^1_u) - v_y(v_{yy}\xi^2_v + u_{yy}\xi^2_u) - v_z(v_{yy}\xi^3_v + u_{yy}\xi^3_u) \\
&- 2v_{yz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right),
\end{aligned}
\tag{1.17j}
$$

$$
\begin{aligned}
\zeta^2_{yz} =& v_{yz}\eta^1_v + u_{yz}\eta^1_u - v_{xy}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) - v_{yy}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - v_{yz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) \\
&+ v_y\left(v_z\eta^1_{vv} + u_z\eta^1_{uv} + \eta^1_{zv}\right) + u_y\left(v_z\eta^1_{uv} + u_z\eta^1_{uu} + \eta^1_{zu}\right) - v_{xz}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) \\
&+ \eta^1_{yz} - v_x\left(v_{yz}\xi^1_v + u_{yz}\xi^1_u + v_y\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) + u_y\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right)\right) \\
&- v_y\left(v_{yz}\xi^2_v + u_{yz}\xi^2_u + v_y\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) + \ v_z\xi^2_{yv} + u_z\xi^2_{yu} + \xi^2_{yz}\right) \\
&- v_z\left(v_{yz}\xi^3_v + u_{yz}\xi^3_u + v_y\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + \ v_z\xi^3_{yv} + u_z\xi^3_{yu} + \xi^3_{yz}\right) \\
&- v_{zz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) + v_z\eta^1_{yv} + u_z\eta^1_{yu} - v_{yz}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) \\
&+ v_y(v_z\xi^1_{yv} + u_z\xi^1_{yu} + \xi^1_{yz}) - v_z u_y\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) - v_y u_y\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right),
\end{aligned}
\tag{1.17k}
$$

$$
\begin{aligned}
\zeta^2_{zz} =& v_{zz}\eta^2_v + u_{zz}\eta^2_u - 2v_{yz}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - 2v_{zz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^2_{zv} \\
&+ v_z\left(v_z\eta^2_{vv} + u_z\eta^2_{uv} + \eta^2_{zv}\right) - v_x\left(v_{zz}\xi^1_v + u_{zz}\xi^1_u + v_z\xi^1_{zv}\right) + \eta^2_{zz} - v_y\xi^2_{zz} \\
&- v_x\left(v_z\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) + u_z\xi^1_{zu} + \ u_z\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + \xi^1_{zz}\right) \\
&- v_y\left(v_z\xi^2_{zv} + v_z\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) + u_z\xi^2_{zu} + \ u_z\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right)\right) \\
&- v_z\left(v_z\xi^3_{zv} + v_z\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + \ u_z\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + \xi^3_{zz}\right) \\
&- 2v_{xz}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) + u_z\eta^2_{zu} + u_z\left(v_z\eta^2_{uv} + u_z\eta^2_{uu} + \eta^2_{zu}\right) + v_z + u_z\xi^3_{zu} \\
&- v_y(v_{zz}\xi^2_v + u_{zz}\xi^2_u) - v_z(v_{zz}\xi^3_v + u_{zz}\xi^3_u).
\end{aligned}
\tag{1.17l}
$$

The first prolongation of X defined by

$$
X^{[1]} = X + \zeta^1_x\frac{\partial}{\partial_{u_x}} + \zeta^1_y\frac{\partial}{\partial_{u_y}} + \zeta^1_z\frac{\partial}{\partial_{u_x}} + \zeta^2_x\frac{\partial}{\partial_{v_x}} + \zeta^2_y\frac{\partial}{\partial_{v_y}} + \zeta^2_z\frac{\partial}{\partial_{v_x}}. \tag{1.18}
$$

The coefficients $\zeta' s$ are

$$
\zeta^1_x = v_x\eta^1_v + u_x\eta^1_u + \eta^1_x - u_x\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - u_y\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - u_z\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right), \tag{1.19a}
$$

$$\zeta^1_y = v_y\eta^1_v + u_y\eta^1_u + \eta^1_y - u_x\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - u_y\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) - u_z\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right), \quad (1.19b)$$

$$\zeta^1_z = v_z\eta^1_v + u_z\eta^1_u + \eta^1_z - u_x\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) - u_y\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - u_z\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right), \quad (1.19c)$$

$$\zeta^2_x = v_x\eta^2_v + u_x\eta^2_u + \eta^2_x - v_x\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - v_y\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - v_z\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right), \quad (1.19d)$$

$$\zeta^2_y = v_y\eta^2_v + u_y\eta^2_u + \eta^2_y - v_x\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - v_y\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) - v_z\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right), \quad (1.19e)$$

$$\zeta^2_z = v_z\eta^2_v + u_z\eta^2_u + \eta^2_z - v_x\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right) - v_y\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - v_z\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right). \quad (1.19f)$$

The expanded form of Eq. (1.15) consists of ξ's, η's, dependent variables and their derivatives. Separation by powers and products of derivatives of u and v yields an overdetermined system of linear homogeneous PDEs in $\xi^{1,2,3}, \eta^{1,2}$. The solution of this system gives the coefficients of symmetry generator, ξ's and η's.

In Chapter 2, symmetry analysis for the eight systems of elliptic equations is presented. Chapter 3 presents the optimal system which is used for similarity reductions and the derivation of invariant solutions. Chapter 4 deals with the conservation laws.

Chapter 2

Symmetries of Elliptic Systems

In this Chapter we calculate symmetries of each of the eight systems introduced in section 1.1.

2.1 System 1

The system (1.6a) in expanded form reads

$$u_{xx} + u_{yy} + u_{zz} = u, \tag{2.1a}$$
$$v_{xx} + v_{yy} + v_{zz} = \alpha v, \quad \alpha \neq 1. \tag{2.1b}$$

It is a second order system of PDEs and hence it admits the one-parameter Lie group of transformations with the generator of symmetries (1.14) provided

$$X^{[2]}\big(u_{xx} + u_{yy} + u_{zz} - u\big)|_{(2.1a)} = 0, \tag{2.2a}$$
$$X^{[2]}\big(v_{xx} + v_{yy} + v_{zz} - \alpha v\big)|_{(2.1b)} = 0, \tag{2.2b}$$

where $X^{[2]}$ is the second prolongation of X given by (1.16). It can be seen from (1.16) that the contributing terms in (2.2a) involve ζ^1_{xx}, ζ^1_{yy}, ζ^1_{zz}, η^1 and ζ^2_{xx}, ζ^2_{yy} , ζ^2_{zz}, η^2 in (2.2b). Upon acting on system (2.1) by $X^{[2]}$, i.e., from (2.2a) and (2.2b) we get the following system

$$\zeta^1_{xx} + \zeta^1_{yy} + \zeta^1_{zz} - \eta^1 = 0, \tag{2.3a}$$
$$\zeta^2_{xx} + \zeta^2_{yy} + \zeta^2_{zz} - \alpha\eta^2 = 0. \tag{2.3b}$$

The expanded form of system (2.3) reads

$$
\begin{aligned}
&-\eta^1 + v_{zz}\eta^1_v + v_{yy}\eta^1_v + v_{xx}\eta^1_v + u_{zz}\eta^1_u + u_{yy}\eta^1_u + u_{xx}\eta^1_u - 2u_{xz}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right)\\
&-2u_{yz}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - 2u_{zz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^1_{zv} + v_z\left(v_z\eta^1_{vv} + u_z\eta^1_{uv} + \eta^1_{zv}\right)\\
&+u_z\left(v_z\eta^1_{uv} + u_z\eta^1_{uu} + \eta^1_{zu}\right) + \eta^1_{zz} + v_y\left(v_y\eta^1_{vv} + u_y\eta^1_{uv} + \eta^1_{yv}\right) + u_y\eta^1_{yu} + u_y\left(v_y\eta^1_{uv} + u_y\eta^1_{uu} + \eta^1_{yu}\right)\\
&-u_x\left(v_{zz}\xi^1_v + u_{zz}\xi^1_u + v_z\xi^1_{zv} + v_z\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) + u_z\xi^1_{zu} + u_z\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + \xi^1_{zz}\right)\\
&-u_y\left(v_{zz}\xi^2_v + u_{zz}\xi^2_u + v_z\xi^2_{zv} + v_z\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) + u_z\xi^2_{zu} + u_z\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) + \xi^2_{zz}\right)\\
&-u_z\left(v_{zz}\xi^3_v + u_{zz}\xi^3_u + v_z\xi^3_{zv} + v_z\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + u_z\xi^3_{zu} + u_z\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + \xi^3_{zz}\right)\\
&-2u_{xy}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - 2u_{yy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) - 2u_{yz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) + v_y\eta^1_{yv} + \eta^1_{yy}\\
&-u_x\left(v_{yy}\xi^1_v + u_{yy}\xi^1_u + v_y\xi^1_{yv} + v_y\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) + u_y\xi^1_{yu} + u_y\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) + \xi^1_{yy}\right)\\
&-u_y\left(v_{yy}\xi^2_v + u_{yy}\xi^2_u + v_y\xi^2_{yv} + v_y\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) + u_y\xi^2_{yu} + u_y\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) + \xi^2_{yy}\right)\\
&-u_z\left(v_{yy}\xi^3_v + u_{yy}\xi^3_u + v_y\xi^3_{yv} + v_y\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right) + u_y\xi^3_{yu} + u_y\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + \xi^3_{yy}\right)\\
&-2u_{xx}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - 2u_{xy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - 2u_{xz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) + v_x\eta^1_{xv}\\
&-u_x\left(v_{xx}\xi^1_v + u_{xx}\xi^1_u + v_x\xi^1_{xv} + v_x\left(v_x\xi^1_{vv} + u_x\xi^1_{uv} + \xi^1_{xv}\right) + u_x\xi^1_{xu} + u_x\left(v_x\xi^1_{uv} + u_x\xi^1_{uu} + \xi^1_{xu}\right) + \xi^1_{xx}\right)\\
&-u_y\left(v_{xx}\xi^2_v + u_{xx}\xi^2_u + v_x\xi^2_{xv} + v_x\left(v_x\xi^2_{vv} + u_x\xi^2_{uv} + \xi^2_{xv}\right) + u_x\xi^2_{xu} + u_x\left(v_x\xi^2_{uv} + u_x\xi^2_{uu} + \xi^2_{xu}\right) + \xi^2_{xx}\right)\\
&-u_z\left(v_{xx}\xi^3_v + u_{xx}\xi^3_u + v_x\xi^3_{xv} + v_x\left(v_x\xi^3_{vv} + u_x\xi^3_{uv} + \xi^3_{xv}\right) + u_x\xi^3_{xu} + u_x\left(v_x\xi^3_{uv} + u_x\xi^3_{uu} + \xi^3_{xu}\right) + \xi^3_{xx}\right)\\
&+v_x\left(v_x\eta^1_{vv} + u_x\eta^1_{uv} + \eta^1_{xv}\right) + u_x\eta^1_{xu} + u_x\left(v_x\eta^1_{uv} + u_x\eta^1_{uu} + \eta^1_{xu}\right) + \eta^1_{xx} + u_z\eta^1_{zu} = 0,
\end{aligned}
\tag{2.4a}
$$

$$
\begin{aligned}
&-\alpha\eta^2 + v_{zz}\eta^2_v + v_{yy}\eta^2_v + v_{xx}\eta^2_v + u_{zz}\eta^2_u + u_{yy}\eta^2_u + u_{xx}\eta^2_u - 2v_{xz}\left(v_z\xi^1_v + u_z\xi^1_u + \xi^1_z\right)\\
&-2v_{yz}\left(v_z\xi^2_v + u_z\xi^2_u + \xi^2_z\right) - 2v_{zz}\left(v_z\xi^3_v + u_z\xi^3_u + \xi^3_z\right) + v_z\eta^2_{zv} + v_z\left(v_z\eta^2_{vv} + u_z\eta^2_{uv} + \eta^2_{zv}\right) + u_z\eta^2_{zu}\\
&+u_z\left(v_z\eta^2_{uv} + u_z\eta^2_{uu} + \eta^2_{zu}\right) + \eta^2_{zz} - v_x\left(v_{zz}\xi^1_v + u_{zz}\xi^1_u + v_z\xi^1_{zv} + v_z\left(v_z\xi^1_{vv} + u_z\xi^1_{uv} + \xi^1_{zv}\right) + u_z\xi^1_{zu}\right.\\
&\left.+u_z\left(v_z\xi^1_{uv} + u_z\xi^1_{uu} + \xi^1_{zu}\right) + \xi^1_{zz}\right) - v_y\left(v_{zz}\xi^2_v + u_{zz}\xi^2_u + v_z\xi^2_{zv} + v_z\left(v_z\xi^2_{vv} + u_z\xi^2_{uv} + \xi^2_{zv}\right) + u_z\xi^2_{zu}\right.\\
&\left.+u_z\left(v_z\xi^2_{uv} + u_z\xi^2_{uu} + \xi^2_{zu}\right) + \xi^2_{zz}\right) - v_z\left(v_{zz}\xi^3_v + u_{zz}\xi^3_u + v_z\xi^3_{zv} + v_z\left(v_z\xi^3_{vv} + u_z\xi^3_{uv} + \xi^3_{zv}\right) + u_z\xi^3_{zu}\right.\\
&\left.+u_z\left(v_z\xi^3_{uv} + u_z\xi^3_{uu} + \xi^3_{zu}\right) + \xi^3_{zz}\right) - 2v_{xy}\left(v_y\xi^1_v + u_y\xi^1_u + \xi^1_y\right) - 2v_{yy}\left(v_y\xi^2_v + u_y\xi^2_u + \xi^2_y\right) + \eta^2_{yy}\\
&-2v_{yz}\left(v_y\xi^3_v + u_y\xi^3_u + \xi^3_y\right) + v_y\eta^2_{yv} + v_y\left(v_y\eta^2_{vv} + u_y\eta^2_{uv} + \eta^2_{yv}\right) + u_y\eta^2_{yu} + u_y\left(v_y\eta^2_{uv} + u_y\eta^2_{uu} + \eta^2_{yu}\right)\\
&-v_x\left(v_{yy}\xi^1_v + u_{yy}\xi^1_u + v_y\xi^1_{yv} + v_y\left(v_y\xi^1_{vv} + u_y\xi^1_{uv} + \xi^1_{yv}\right) + u_y\xi^1_{yu} + u_y\left(v_y\xi^1_{uv} + u_y\xi^1_{uu} + \xi^1_{yu}\right) + \xi^1_{yy}\right)\\
&-v_y\left(v_{yy}\xi^2_v + u_{yy}\xi^2_u + v_y\xi^2_{yv} + v_y\left(v_y\xi^2_{vv} + u_y\xi^2_{uv} + \xi^2_{yv}\right) + u_y\xi^2_{yu} + u_y\left(v_y\xi^2_{uv} + u_y\xi^2_{uu} + \xi^2_{yu}\right) + \xi^2_{yy}\right)\\
&-v_z\left(v_{yy}\xi^3_v + u_{yy}\xi^3_u + v_y\xi^3_{yv} + v_y\left(v_y\xi^3_{vv} + u_y\xi^3_{uv} + \xi^3_{yv}\right) + u_y\xi^3_{yu} + u_y\left(v_y\xi^3_{uv} + u_y\xi^3_{uu} + \xi^3_{yu}\right) + \xi^3_{yy}\right)\\
&-2v_{xx}\left(v_x\xi^1_v + u_x\xi^1_u + \xi^1_x\right) - 2v_{xy}\left(v_x\xi^2_v + u_x\xi^2_u + \xi^2_x\right) - 2v_{xz}\left(v_x\xi^3_v + u_x\xi^3_u + \xi^3_x\right) + v_x\eta^2_{xv}\\
&+v_x\left(v_x\eta^2_{vv} + u_x\eta^2_{uv} + \eta^2_{xv}\right) + u_x\eta^2_{xu} + u_x\left(v_x\eta^2_{uv} + u_x\eta^2_{uu} + \eta^2_{xu}\right) + \eta^2_{xx} - v_x\left(v_{xx}\xi^1_v + u_{xx}\xi^1_u + v_x\xi^1_{xv}\right.\\
&\left.+v_x\left(v_x\xi^1_{vv} + u_x\xi^1_{uv} + \xi^1_{xv}\right) + u_x\xi^1_{xu} + u_x\left(v_x\xi^1_{uv} + u_x\xi^1_{uu} + \xi^1_{xu}\right) + \xi^1_{xx}\right) - v_y\left(v_{xx}\xi^2_v + u_{xx}\xi^2_u + v_x\xi^2_{xv}\right.\\
&\left.+v_x\left(v_x\xi^2_{vv} + u_x\xi^2_{uv} + \xi^2_{xv}\right) + u_x\xi^2_{xu} + u_x\left(v_x\xi^2_{uv} + u_x\xi^2_{uu} + \xi^2_{xu}\right) + \xi^2_{xx}\right) - v_z\left(v_{xx}\xi^3_v + u_{xx}\xi^3_u + v_x\xi^3_{xv}\right.\\
&\left.+v_x\left(v_x\xi^3_{vv} + u_x\xi^3_{uv} + \xi^3_{xv}\right) + u_x\xi^3_{xu} + u_x\left(v_x\xi^3_{uv} + u_x\xi^3_{uu} + \xi^3_{xu}\right) + \xi^3_{xx}\right) = 0.
\end{aligned}
\tag{2.4b}
$$

Using the fact that $\xi^1, \xi^2, \xi^3, \eta^1$ and η^2 do not depend on the derivatives of u and v, separating (2.4) by powers and products of derivatives of u and v yields the following list of determining equations

$$\xi^1_v = 0, \tag{2.5a}$$
$$\xi^2_v = 0, \tag{2.5b}$$
$$\xi^3_v = 0, \tag{2.5c}$$
$$\eta^1_{vv} = 0, \tag{2.5d}$$
$$\eta^2_{vv} = 0, \tag{2.5e}$$
$$\xi^1_u = 0, \tag{2.5f}$$
$$\xi^2_u = 0, \tag{2.5g}$$
$$\xi^3_u = 0, \tag{2.5h}$$
$$\eta^1_{uv} = 0, \tag{2.5i}$$
$$\eta^2_{uv} = 0, \tag{2.5j}$$
$$\eta^1_{uu} = 0, \tag{2.5k}$$
$$\eta^2_{uu} = 0, \tag{2.5l}$$
$$\eta^1_{zv} = 0, \tag{2.5m}$$
$$\eta^2_{zu} = 0, \tag{2.5n}$$
$$\eta^1_{yv} = 0, \tag{2.5o}$$
$$\eta^2_{yu} = 0, \tag{2.5p}$$
$$\eta^1_{xv} = 0, \tag{2.5q}$$
$$\eta^2_{xu} = 0, \tag{2.5r}$$
$$2\xi^2_z + 2\xi^3_y = 0, \tag{2.5s}$$
$$2\xi^3_z - 2\xi^1_x = 0, \tag{2.5t}$$
$$2\xi^2_y - 2\xi^1_x = 0, \tag{2.5u}$$
$$2\xi^1_y + 2\xi^2_x = 0, \tag{2.5v}$$
$$2\xi^1_z + 2\xi^3_x = 0, \tag{2.5w}$$
$$v\alpha\eta^1_v + u\eta^1_u + \eta^1_{zz} + \eta^1_{yy} - 2u\xi^1_x + \eta^1_{xx} - \eta^1 = 0, \tag{2.5x}$$
$$-\xi^1_{zz} - \xi^1_{yy} + 2\eta^1_{xu} - \xi^1_{xx} = 0, \tag{2.5y}$$
$$-\xi^2_{zz} + 2\eta^1_{yu} - \xi^2_{yy} - \xi^2_{xx} = 0, \tag{2.5z}$$
$$2\eta^1_{zu} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0, \tag{2.5aa}$$
$$2\xi^3_z - 2\xi^2_y = 0, \tag{2.5ab}$$
$$2\xi^2_z + 2\xi^3_y = 0, \tag{2.5ac}$$
$$2\xi^2_y - 2\xi^1_x = 0, \tag{2.5ad}$$
$$2\xi^1_y + 2\xi^2_x = 0, \tag{2.5ae}$$
$$2\xi^1_z + 2\xi^3_x = 0, \tag{2.5af}$$
$$v\alpha\eta^2_v + u\eta^2_u + \eta^2_{zz} - 2v\alpha\xi^2_y + \eta^2_{yy} + \eta^2_{xx} - \alpha\eta^2 = 0, \tag{2.5ag}$$
$$2\eta^2_{xv} - \xi^1_{zz} - \xi^1_{yy} - \xi^1_{xx} = 0, \tag{2.5ah}$$
$$2\eta^2_{yv} - \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} = 0, \tag{2.5ai}$$
$$2\eta^2_{zv} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0. \tag{2.5aj}$$

Solving (2.5a to r) leads to

$$\xi^1 = \xi^1(x,y,z), \tag{2.6a}$$
$$\xi^2 = \xi^2(x,y,z), \tag{2.6b}$$
$$\xi^3 = \xi^3(x,y,z), \tag{2.6c}$$
$$\eta^1 = a(x,y,z) + \frac{u}{2}\left(2K_3 - \xi^2_y\right) + vK_1, \tag{2.6d}$$
$$\eta^2 = d(x,y,z) + uK_2 + \frac{v}{2}\left(2b(x,z) - \xi^1_x\right), \tag{2.6e}$$

where a, b, d are arbitrary functions of the indicated variables and K_1, K_2, K_3 are arbitrary constants. Due to Eqs. (2.6), Eqs. (2.5s to aj) become

$$\xi^2_z + \xi^3_y = 0, \tag{2.7a}$$
$$\xi^3_z - \xi^1_x = 0, \tag{2.7b}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.7c}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.7d}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.7e}$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - \frac{1}{2}\left(2vK_1 + 2a + u\xi^2_{yzz} + u\xi^2_{yyy} + 4u\xi^1_x + u\xi^2_{xxy}\right) = 0, \tag{2.7f}$$
$$\xi^1_{zz} + \xi^1_{yy} + \xi^2_{xy} + \xi^1_{xx} = 0, \tag{2.7g}$$
$$\xi^2_{zz} + 2\xi^2_{yy} + \xi^2_{xx} = 0, \tag{2.7h}$$
$$\xi^3_{zz} + \xi^2_{yz} + \xi^3_{yy} + \xi^3_{xx} = 0, \tag{2.7i}$$
$$\xi^3_z - \xi^2_y = 0, \tag{2.7j}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.7k}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.7l}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.7m}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.7n}$$
$$\begin{aligned} &uK_2 + vb_{zz} + vb_{xx} + d_{zz} + d_{yy} + d_{xx} \\ &-\frac{1}{2}\left(2u\alpha K_2 + 2\alpha d + 4v\alpha\xi^2_y + v\xi^1_{xzz} + v\xi^1_{xyy} + v\xi^1_{xxx}\right) = 0, \end{aligned} \tag{2.7o}$$
$$2b_x + \xi^1_{zz} + \xi^1_{yy} + 2\xi^1_{xx} = 0, \tag{2.7p}$$
$$\xi^2_{zz} + \xi^2_{yy} + \xi^1_{xy} + \xi^2_{xx} = 0, \tag{2.7q}$$
$$2b_z - \xi^3_{zz} - \xi^3_{yy} - \xi^1_{xz} - \xi^3_{xx} = 0. \tag{2.7r}$$

Differentiating (2.7c,l) with respect to x and (2.7d,m) with respect to y yield

$$\xi^2_{xy} - \xi^1_{xx} = 0, \tag{2.8a}$$
$$\xi^1_{yy} + \xi^2_{xy} = 0, \tag{2.8b}$$

respectively. From Eqs. (2.8) we get

$$\xi^1_{xx} + \xi^1_{yy} = 0. \tag{2.9}$$

Similarly differentiating (2.7b) and (2.7e,n) with respect to x and z respectively leads to

$$\xi^3_{xz} - \xi^1_{xx} = 0, \tag{2.10a}$$
$$\xi^3_{xz} + \xi^1_{xx} = 0. \tag{2.10b}$$

From (2.10a) and (2.10b) we get

$$\xi^1_{xx} + \xi^1_{zz} = 0. \tag{2.11}$$

Substituting (2.9) and (2.11) into (2.7p) we obtain

$$b = b(z). \tag{2.12}$$

Differentiating (2.7b) with respect to z and (2.7e,n) with respect to x respectively give

$$\xi^3_{zz} - \xi^1_{xz} = 0, \tag{2.13a}$$
$$\xi^1_{xz} + \xi^3_{xx} = 0, \tag{2.13b}$$

which both imply that

$$\xi^3_{xx} + \xi^3_{zz} = 0. \tag{2.14}$$

From (2.14) and (2.7r) we have

$$-2b_z + \xi^3_{yy} + \xi^1_{xz} = 0. \tag{2.15}$$

Differentiating (2.7a) with respect to y gives

$$\xi^2_{yz} + \xi^3_{yy} = 0. \tag{2.16}$$

From (2.7c,l) it can be concluded that

$$\xi^2_{yz} = \xi^1_{xz}. \tag{2.17}$$

Substitution of (2.17) into (2.16) yields

$$\xi^3_{yy} + \xi^1_{xz} = 0. \tag{2.18}$$

Substituting (2.18) into (2.15) implies that b is a constant, (i.e. $b = K_4$).

Differentiating (2.7f) with respect to u and (2.7o) with respect to v respectively give equations

$$4\xi^1_x + \xi^2_{xxy} + \xi^2_{yyy} + \xi^2_{yzz} = 0, \tag{2.19a}$$
$$4\alpha\xi^2_y + \xi^1_{xxx} + \xi^1_{xyy} + \xi^1_{xzz} = 0. \tag{2.19b}$$

Equations (2.19a,b) used together with (2.7c,l) imply that $\xi^1 = \xi^1(y,z)$, which in turn when used together with (2.7b) and (2.7c) imply that $\xi^3 = \xi^3(x,y)$ and $\xi^2 = \xi^2(x,z)$ respectively. The rest of equations (2.7) become

$$\xi^2_z + \xi^3_y = 0, \tag{2.20a}$$

$$\xi^1_y + \xi^2_x = 0, \tag{2.20b}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.20c}$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - vK_1 - a = 0, \tag{2.20d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.20e}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.20f}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0, \tag{2.20g}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.20h}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.20i}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.20j}$$
$$uK_2 + d_{zz} + d_{yy} + d_{xx} - \alpha(uK_2 - d) = 0, \tag{2.20k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.20l}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.20m}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0. \tag{2.20n}$$

Integrating (2.20a) with respect to y yields:

$$\xi^3 = -y\xi^2_z + e^1(x), \tag{2.21}$$

where e^1 is an arbitrary function of x. Substitution of (2.21) into equations (2.20) yields

$$y\xi^2_{zz} = 0, \tag{2.22a}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.22b}$$
$$e^1_x + \xi^1_z - y\xi^2_{xz} = 0, \tag{2.22c}$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - vK_1 - a = 0, \tag{2.22d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.22e}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.22f}$$
$$-e^1_{xx} + y\left(\xi^2_{zzz} + \xi^2_{xxz}\right) = 0, \tag{2.22g}$$
$$y\xi^2_{zz} = 0, \tag{2.22h}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.22i}$$
$$e^1_x + \xi^1_z - y\xi^2_{xz} = 0, \tag{2.22j}$$
$$uK_2 + d_{zz} + d_{yy} + d_{xx} - \alpha(uK_2 - d) = 0, \tag{2.22k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.22l}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.22m}$$
$$-e^1_{xx} + y\left(\xi^2_{zzz} + \xi^2_{xxz}\right) = 0. \tag{2.22n}$$

Equations (2.22a,h) give

$$\xi^2 = e^2(x) + ze^3(x), \tag{2.23}$$

where e^2 and e^3 are arbitrary functions of x. The remaining equations are

$$e^2_x + ze^3_x + \xi^1_y = 0, \tag{2.24a}$$

$$e^1_x + \xi^1_z - ye^3_x = 0, \quad (2.24b)$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - vK_1 + a = 0, \quad (2.24c)$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \quad (2.24d)$$
$$e^2_{xx} + ze^3_{xx} = 0, \quad (2.24e)$$
$$e^1_{xx} - ye^3_{xx} = 0, \quad (2.24f)$$
$$e^2_x + ze^3_x + \xi^1_y = 0, \quad (2.24g)$$
$$e^1_x + \xi^1_z - ye^3_x = 0, \quad (2.24h)$$
$$uK_2 + d_{zz} + d_{yy} + d_{xx} - \alpha(uK_2 + d) = 0, \quad (2.24i)$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \quad (2.24j)$$
$$e^2_{xx} + ze^3_{xx} = 0, \quad (2.24k)$$
$$e^1_{xx} - ye^3_{xx} = 0. \quad (2.24l)$$

Integrating (2.24a,g) with respect to y give

$$\xi^1 = z(-e^1_x + ye^3_x) + e^4(y), \quad (2.25)$$

for some arbitrary function $e^4(y)$. Substituting (2.25) into equations involving ξ^1 in (2.24) gives

$$e^1 = K_5 + xK_6, \quad (2.26a)$$
$$e^2 = K_7 + xK_8, \quad (2.26b)$$
$$e^3 = K_9 + xK_{10}, \quad (2.26c)$$

where $K_5, ..., K_{10}$ are arbitrary constants. Equations yet to be satisfied in (2.24) are

$$K_8 + 2zK_{10} + e^4_y = 0, \quad (2.27a)$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - vK_1 + a = 0, \quad (2.27b)$$
$$e^4_{yy} = 0, \quad (2.27c)$$
$$K_8 + 2zK_{10} + e^4_y = 0, \quad (2.27d)$$
$$uK_2 + d_{zz} + d_{yy} + d_{xx} - \alpha(uK_1 + d) = 0, \quad (2.27e)$$
$$e^4_{yy} = 0. \quad (2.27f)$$

From (2.27a,d) we find that

$$e^4 = K_{11} - y(K_8 + 2zK_{10}), \quad (2.28)$$

for some arbitrary constant K_{11}. Substitution of (2.28) into equations (2.27) yields

$$yK_{10} = 0, \quad (2.29a)$$
$$v\alpha K_1 + a_{zz} + a_{yy} + a_{xx} - vK_1 + a = 0, \quad (2.29b)$$
$$yK_{10} = 0, \quad (2.29c)$$
$$uK_2 + d_{zz} + d_{yy} + d_{xx} - \alpha(uK_2 + d) = 0. \quad (2.29d)$$

From (2.29a,c), $K_{10} = 0$. Also since $\alpha \neq 1$, $K_1 = K_2 = 0$. Making these substitutions leave us with the equations

$$a_{xx} + a_{yy} + a_{zz} = a, \tag{2.30a}$$
$$d_{xx} + d_{yy} + d_{zz} = \alpha d, \tag{2.30b}$$

which is system (2.1) in new dependent variables a and d. Thus, the coefficients of symmetry generator are

$$\xi^1 = -zK_6 - yK_8 + K_{11}, \tag{2.31a}$$
$$\xi^2 = K_7 + xK_8 + zK_9, \tag{2.31b}$$
$$\xi^3 = K_5 + xK_6 - yK_9, \tag{2.31c}$$
$$\eta^1 = uK_3 + a, \tag{2.31d}$$
$$\eta^2 = vK_4 + d. \tag{2.31e}$$

Therefore from (2.31) the following symmetries are obtained

$$X_1 = \partial_x, \tag{2.32a}$$
$$X_2 = \partial_y, \tag{2.32b}$$
$$X_3 = \partial_z, \tag{2.32c}$$
$$X_4 = y\partial_x - x\partial_y, \tag{2.32d}$$
$$X_5 = z\partial_x - x\partial_z, \tag{2.32e}$$
$$X_6 = z\partial_y - y\partial_z, \tag{2.32f}$$
$$X_7 = u\partial_u, \tag{2.32g}$$
$$X_8 = v\partial_v, \tag{2.32h}$$
$$X_a = a(x, y, z)\partial_u, \tag{2.32i}$$
$$X_d = d(x, y, z)\partial_v, \tag{2.32j}$$

where $a(x, y, z)$ and $d(x, y, z)$ satisfy system (2.1) or (2.30) whenever $\alpha \neq 1$.

In expanded form, system (1.6b) becomes

$$u_{xx} + u_{yy} + u_{zz} = -u, \tag{2.33a}$$
$$v_{xx} + v_{yy} + v_{zz} = \alpha v, \ \alpha \neq 1. \tag{2.33b}$$

This is system 2, the computations have shown that it admits symmetries (2.32).

2.2 System 3

Consider the second order system of PDEs

$$u_{xx} + u_{yy} + u_{zz} = 0, \tag{2.34a}$$
$$v_{xx} + v_{yy} + v_{zz} = 0, \tag{2.34b}$$

which is the expanded form of system (1.6c). The system admits one-parameter Lie group of transformation with the generator given by (1.14) whenever

$$X^{[2]}\left(u_{xx}+u_{yy}+u_{zz}\right)|_{(2.34a)}=0, \tag{2.35a}$$

$$X^{[2]}\left(v_{xx}+v_{yy}+v_{zz}\right)|_{(2.34b)}=0, \tag{2.35b}$$

where $X^{[2]}$ is the second prolongation of X given by (1.16). Expanding (2.35) yields the following equations

$$\begin{aligned}
&v_{zz}\eta^1_v+v_{yy}\eta^1_v+v_{xx}\eta^1_v+u_{zz}\eta^1_u+u_{yy}\eta^1_u+u_{xx}\eta^1_u-2u_{xz}\left(v_z\xi^1_v+u_z\xi^1_u+\xi^1_z\right)-2u_{yz}\left(v_z\xi^2_v+u_z\xi^2_u+\xi^2_z\right)\\
&-2u_{zz}\left(v_z\xi^3_v+u_z\xi^3_u+\xi^3_z\right)+v_z\eta^1_{zv}+v_z\left(v_z\eta^1_{vv}+u_z\eta^1_{uv}+\eta^1_{zv}\right)+u_z\eta^1_{zu}+u_z\left(v_z\eta^1_{uv}+u_z\eta^1_{uu}+\eta^1_{zu}\right)\\
&+\eta^1_{zz}-u_x\left(v_{zz}\xi^1_v+u_{zz}\xi^1_u+v_z\xi^1_{zv}+v_z\left(v_z\xi^1_{vv}+u_z\xi^1_{uv}+\xi^1_{zv}\right)+u_z\xi^1_{zu}+u_z\left(v_z\xi^1_{uv}+u_z\xi^1_{uu}+\xi^1_{zu}\right)\right)\\
&-u_y\left(v_{zz}\xi^2_v+u_{zz}\xi^2_u+v_z\xi^2_{zv}+v_z\left(v_z\xi^2_{vv}+u_z\xi^2_{uv}+\xi^2_{zv}\right)+u_z\xi^2_{zu}+u_z\left(v_z\xi^2_{uv}+u_z\xi^2_{uu}+\xi^2_{zu}\right)+\xi^2_{zz}\right)\\
&-u_z\left(v_{zz}\xi^3_v+u_{zz}\xi^3_u+v_z\xi^3_{zv}+v_z\left(v_z\xi^3_{vv}+u_z\xi^3_{uv}+\xi^3_{zv}\right)+u_z\xi^3_{zu}+u_z\left(v_z\xi^3_{uv}+u_z\xi^3_{uu}+\xi^3_{zu}\right)+\xi^3_{zz}\right)\\
&-2u_{yy}\left(v_y\xi^2_v+u_y\xi^2_u+\xi^2_y\right)-2u_{yz}\left(v_y\xi^3_v+u_y\xi^3_u+\xi^3_y\right)+v_y\eta^1_{yv}+v_x\left(v_x\eta^1_{vv}+u_x\eta^1_{uv}+\eta^1_{xv}\right)+\eta^1_{xx}\\
&-2u_{xy}\left(v_y\xi^1_v+u_y\xi^1_u+\xi^1_y\right)+v_y\left(v_y\eta^1_{vv}+u_y\eta^1_{uv}+\eta^1_{yv}\right)+u_y\eta^1_{yu}+u_y\left(v_y\eta^1_{uv}+u_y\eta^1_{uu}+\eta^1_{yu}\right)+\eta^1_{yy}\\
&-u_x\left(v_{yy}\xi^1_v+u_{yy}\xi^1_u+v_y\xi^1_{yv}+v_y\left(v_y\xi^1_{vv}+u_y\xi^1_{uv}+\xi^1_{yv}\right)+u_y\xi^1_{yu}+u_y\left(v_y\xi^1_{uv}+u_y\xi^1_{uu}+\xi^1_{yu}\right)+\xi^1_{yy}\right)\\
&-u_y\left(v_{yy}\xi^2_v+u_{yy}\xi^2_u+v_y\xi^2_{yv}+v_y\left(v_y\xi^2_{vv}+u_y\xi^2_{uv}+\xi^2_{yv}\right)+u_y\xi^2_{yu}+u_y\left(v_y\xi^2_{uv}+u_y\xi^2_{uu}+\xi^2_{yu}\right)+\xi^2_{yy}\right)\\
&-u_z\left(v_{yy}\xi^3_v+u_{yy}\xi^3_u+v_y\xi^3_{yv}+v_y\left(v_y\xi^3_{vv}+u_y\xi^3_{uv}+\xi^3_{yv}\right)+u_y\xi^3_{yu}+u_y\left(v_y\xi^3_{uv}+u_y\xi^3_{uu}+\xi^3_{yu}\right)+\xi^3_{yy}\right)\\
&-2u_{xx}\left(v_x\xi^1_v+u_x\xi^1_u+\xi^1_x\right)-2u_{xy}\left(v_x\xi^2_v+u_x\xi^2_u+\xi^2_x\right)-2u_{xz}\left(v_x\xi^3_v+u_x\xi^3_u+\xi^3_x\right)+v_x\eta^1_{xv}+u_x\eta^1_{xu}\\
&-u_x\left(v_{xx}\xi^1_v+u_{xx}\xi^1_u+v_x\xi^1_{xv}+v_x\left(v_x\xi^1_{vv}+u_x\xi^1_{uv}+\xi^1_{xv}\right)+u_x\xi^1_{xu}+u_x\left(v_x\xi^1_{uv}+u_x\xi^1_{uu}+\xi^1_{xu}\right)+\xi^1_{xx}\right)\\
&-u_y\left(v_{xx}\xi^2_v+u_{xx}\xi^2_u+v_x\xi^2_{xv}+v_x\left(v_x\xi^2_{vv}+u_x\xi^2_{uv}+\xi^2_{xv}\right)+u_x\xi^2_{xu}+u_x\left(v_x\xi^2_{uv}+u_x\xi^2_{uu}+\xi^2_{xu}\right)+\xi^2_{xx}\right)\\
&-u_z\left(v_{xx}\xi^3_v+u_{xx}\xi^3_u+v_x\xi^3_{xv}+v_x\left(v_x\xi^3_{vv}+u_x\xi^3_{uv}+\xi^3_{xv}\right)+u_x\xi^3_{xu}+u_x\left(v_x\xi^3_{uv}+u_x\xi^3_{uu}+\xi^3_{xu}\right)+\xi^3_{xx}\right)\\
&+u_x\left(v_x\eta^1_{uv}+u_x\eta^1_{uu}+\eta^1_{xu}\right)-u_x\xi^1_{zz}=0,
\end{aligned} \tag{2.36a}$$

$$\begin{aligned}
&v_{zz}\eta^2_v+v_{yy}\eta^2_v+v_{xx}\eta^2_v+u_{zz}\eta^2_u+u_{yy}\eta^2_u+u_{xx}\eta^2_u-2v_{xz}\left(v_z\xi^1_v+u_z\xi^1_u+\xi^1_z\right)-2v_{yz}\left(v_z\xi^2_v+u_z\xi^2_u+\xi^2_z\right)\\
&-2v_{zz}\left(v_z\xi^3_v+u_z\xi^3_u+\xi^3_z\right)+v_z\eta^2_{zv}+v_z\left(v_z\eta^2_{vv}+u_z\eta^2_{uv}+\eta^2_{zv}\right)+u_z\eta^2_{zu}+u_z\left(v_z\eta^2_{uv}+u_z\eta^2_{uu}+\eta^2_{zu}\right)\\
&-v_x\left(v_{zz}\xi^1_v+u_{zz}\xi^1_u+v_z\xi^1_{zv}+v_z\left(v_z\xi^1_{vv}+u_z\xi^1_{uv}+\xi^1_{zv}\right)+u_z\xi^1_{zu}+u_z\left(v_z\xi^1_{uv}+u_z\xi^1_{uu}+\xi^1_{zu}\right)+\xi^1_{zz}\right)\\
&-v_y\left(v_{zz}\xi^2_v+u_{zz}\xi^2_u+v_z\xi^2_{zv}+v_z\left(v_z\xi^2_{vv}+u_z\xi^2_{uv}+\xi^2_{zv}\right)+u_z\xi^2_{zu}+u_z\left(v_z\xi^2_{uv}+u_z\xi^2_{uu}+\xi^2_{zu}\right)+\xi^2_{zz}\right)\\
&-v_z\left(v_{zz}\xi^3_v+u_{zz}\xi^3_u+v_z\xi^3_{zv}+v_z\left(v_z\xi^3_{vv}+u_z\xi^3_{uv}+\xi^3_{zv}\right)+u_z\xi^3_{zu}+u_z\left(v_z\xi^3_{uv}+u_z\xi^3_{uu}+\xi^3_{zu}\right)+\xi^3_{zz}\right)\\
&-2v_{xy}\left(v_y\xi^1_v+u_y\xi^1_u+\xi^1_y\right)-2v_{yy}\left(v_y\xi^2_v+u_y\xi^2_u+\xi^2_y\right)-2v_{yz}\left(v_y\xi^3_v+u_y\xi^3_u+\xi^3_y\right)+v_y\eta^2_{yv}+\eta^2_{zz}\\
&+v_y\left(v_y\eta^2_{vv}+u_y\eta^2_{uv}+\eta^2_{yv}\right)+u_y\eta^2_{yu}+u_y\left(v_y\eta^2_{uv}+u_y\eta^2_{uu}+\eta^2_{yu}\right)+\eta^2_{yy}-v_x\left(v_{yy}\xi^1_v+u_{yy}\xi^1_u+v_y\xi^1_{yv}\right.\\
&\left.+v_y\left(v_y\xi^1_{vv}+u_y\xi^1_{uv}+\xi^1_{yv}\right)+u_y\xi^1_{yu}+u_y\left(v_y\xi^1_{uv}+u_y\xi^1_{uu}+\xi^1_{yu}\right)+\xi^1_{yy}\right)-v_y\left(v_{yy}\xi^2_v+u_{yy}\xi^2_u+v_y\xi^2_{yv}\right.\\
&\left.+v_y\left(v_y\xi^2_{vv}+u_y\xi^2_{uv}+\xi^2_{yv}\right)+u_y\xi^2_{yu}+u_y\left(v_y\xi^2_{uv}+u_y\xi^2_{uu}+\xi^2_{yu}\right)+\xi^2_{yy}\right)-v_z\left(v_{yy}\xi^3_v+u_{yy}\xi^3_u+v_y\xi^3_{yv}\right.\\
&\left.+v_y\left(v_y\xi^3_{vv}+u_y\xi^3_{uv}+\xi^3_{yv}\right)+u_y\xi^3_{yu}+u_y\left(v_y\xi^3_{uv}+u_y\xi^3_{uu}+\xi^3_{yu}\right)+\xi^3_{yy}\right)-2v_{xx}\left(v_x\xi^1_v+u_x\xi^1_u+\xi^1_x\right)\\
&-2v_{xy}\left(v_x\xi^2_v+u_x\xi^2_u+\xi^2_x\right)-2v_{xz}\left(v_x\xi^3_v+u_x\xi^3_u+\xi^3_x\right)+v_x\eta^2_{xv}+v_x\left(v_x\eta^2_{vv}+u_x\eta^2_{uv}+\eta^2_{xv}\right)+u_x\eta^2_{xu}\\
&+u_x\left(v_x\eta^2_{uv}+u_x\eta^2_{uu}+\eta^2_{xu}\right)+\eta^2_{xx}-v_x\left(v_{xx}\xi^1_v+u_{xx}\xi^1_u+v_x\xi^1_{xv}+v_x\left(v_x\xi^1_{vv}+u_x\xi^1_{uv}+\xi^1_{xv}\right)+u_x\xi^1_{xu}\right.\\
&\left.+u_x\left(v_x\xi^1_{uv}+u_x\xi^1_{uu}+\xi^1_{xu}\right)+\xi^1_{xx}\right)-v_y\left(v_{xx}\xi^2_v+u_{xx}\xi^2_u+v_x\xi^2_{xv}+v_x\left(v_x\xi^2_{vv}+u_x\xi^2_{uv}+\xi^2_{xv}\right)+u_x\xi^2_{xu}\right.\\
&\left.+u_x\left(v_x\xi^2_{uv}+u_x\xi^2_{uu}+\xi^2_{xu}\right)+\xi^2_{xx}\right)-v_z\left(v_{xx}\xi^3_v+u_{xx}\xi^3_u+v_x\xi^3_{xv}+v_x\left(v_x\xi^3_{vv}+u_x\xi^3_{uv}+\xi^3_{xv}\right)+u_x\xi^3_{xu}\right.\\
&\left.+u_x\left(v_x\xi^3_{uv}+u_x\xi^3_{uu}+\xi^3_{xu}\right)+\xi^3_{xx}\right)=0.
\end{aligned} \tag{2.36b}$$

Using the same argument from the previous section that $\xi^1, \xi^2, \xi^3, \eta^1$ and η^2 do not depend on the derivatives of u and v, separating (2.36) by powers and products of derivatives of u and v results into the following determining equations

$$\xi^1_v = 0, \tag{2.37a}$$
$$\xi^2_v = 0, \tag{2.37b}$$
$$\xi^3_v = 0, \tag{2.37c}$$
$$\eta^1_{vv} = 0, \tag{2.37d}$$
$$\eta^2_{vv} = 0, \tag{2.37e}$$
$$\xi^1_u = 0, \tag{2.37f}$$
$$\xi^2_u = 0, \tag{2.37g}$$
$$\xi^3_u = 0, \tag{2.37h}$$
$$\eta^1_{uv} = 0, \tag{2.37i}$$
$$\eta^2_{uv} = 0, \tag{2.37j}$$
$$\eta^1_{uu} = 0, \tag{2.37k}$$
$$\eta^2_{uu} = 0, \tag{2.37l}$$
$$\eta^1_{zv} = 0, \tag{2.37m}$$
$$\eta^2_{zu} = 0, \tag{2.37n}$$
$$\eta^1_{yv} = 0, \tag{2.37o}$$
$$\eta^2_{yu} = 0, \tag{2.37p}$$
$$\eta^1_{xv} = 0, \tag{2.37q}$$
$$\eta^2_{xu} = 0, \tag{2.37r}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.37s}$$
$$\xi^3_z - \xi^1_x = 0, \tag{2.37t}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.37u}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.37v}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.37w}$$
$$\eta^1_{zz} + \eta^1_{yy} + \eta^1_{xx} = 0, \tag{2.37x}$$
$$2\eta^1_{xu} - \xi^1_{zz} - \xi^1_{yy} - \xi^1_{xx} = 0, \tag{2.37y}$$
$$2\eta^1_{yu} - \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} = 0, \tag{2.37z}$$
$$2\eta^1_{zu} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0, \tag{2.37aa}$$
$$\xi^3_z - \xi^2_y = 0, \tag{2.37ab}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.37ac}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.37ad}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.37ae}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.37af}$$

$$\eta^2_{zz}+\eta^2_{yy}+\eta^2_{xx}=0, \tag{2.37ag}$$
$$2\eta^2_{xv}-\xi^1_{zz}-\xi^1_{yy}-\xi^1_{xx}=0, \tag{2.37ah}$$
$$2\eta^2_{yv}-\xi^2_{zz}-\xi^2_{yy}-\xi^2_{xx}=0, \tag{2.37ai}$$
$$2\eta^2_{zv}-\xi^3_{zz}-\xi^3_{yy}-\xi^3_{xx}=0. \tag{2.37aj}$$

Solving (2.37a) to (2.37r) we get the following solutions

$$\xi^1=\xi^1(x,y,z), \tag{2.38a}$$
$$\xi^2=\xi^2(x,y,z), \tag{2.38b}$$
$$\xi^3=\xi^3(x,y,z), \tag{2.38c}$$
$$\eta^1=a(x,y,z)+\frac{u}{2}\left(2K_3-\xi^2_y\right)+vK_1, \tag{2.38d}$$
$$\eta^2=d(x,y,z)+uK_2+\frac{v}{2}\left(2K_4-\xi^1_x\right), \tag{2.38e}$$

where a and d are arbitrary functions of their arguments and K_1,K_2,K_3 and K_4 are arbitrary constants.Thus, equations (2.38) transform the rest of the equations from (2.37s) to (2.37aj) :

$$\xi^2_z+\xi^3_y=0, \tag{2.39a}$$
$$\xi^3_z-\xi^1_x=0, \tag{2.39b}$$
$$\xi^2_y-\xi^1_x=0, \tag{2.39c}$$
$$\xi^1_y+\xi^2_x=0, \tag{2.39d}$$
$$\xi^1_z+\xi^3_x=0, \tag{2.39e}$$
$$a_{zz}+a_{yy}+a_{xx}-\frac{1}{2}u\left(\xi^2_{yzz}+\xi^2_{yyy}+\xi^2_{xxy}\right)=0, \tag{2.39f}$$
$$\xi^1_{zz}+\xi^1_{yy}+\xi^2_{xy}+\xi^1_{xx}=0, \tag{2.39g}$$
$$\xi^2_{zz}+2\xi^2_{yy}+\xi^2_{xx}=0, \tag{2.39h}$$
$$\xi^3_{zz}+\xi^2_{yz}+\xi^3_{yy}+\xi^3_{xx}=0, \tag{2.39i}$$
$$\xi^3_z-\xi^2_y=0, \tag{2.39j}$$
$$\xi^2_z+\xi^3_y=0, \tag{2.39k}$$
$$\xi^2_y-\xi^1_x=0, \tag{2.39l}$$
$$\xi^1_y+\xi^2_x=0, \tag{2.39m}$$
$$\xi^1_z+\xi^3_x=0, \tag{2.39n}$$
$$d_{zz}+d_{yy}+d_{xx}-\frac{1}{2}v\left(\xi^1_{xzz}+\xi^1_{xyy}+\xi^1_{xxx}\right)=0, \tag{2.39o}$$
$$\xi^1_{zz}+\xi^1_{yy}+2\xi^1_{xx}=0, \tag{2.39p}$$
$$\xi^2_{zz}+\xi^2_{yy}+\xi^1_{xy}+\xi^2_{xx}=0, \tag{2.39q}$$
$$\xi^3_{zz}+\xi^3_{yy}+\xi^1_{xz}+\xi^3_{xx}=0, \tag{2.39r}$$

Differentiating (2.39o) with respect to v we get

$$\xi^1_{xxx}+\xi^1_{xyy}+\xi^1_{xzz}=0. \tag{2.40}$$

Integrating (2.40) with respect to x we yields

$$\xi^1_{xx}+\xi^1_{yy}+\xi^1_{zz}=F^1(y,z), \tag{2.41}$$

where $F^1(y,z)$ is an arbitrary function. Using (2.39p) together with (2.41) gives

$$\xi^1_{xx} + F^1(y,z) = 0, \tag{2.42}$$

which implies that

$$\xi^1 = F^2(y,z) + xF^3(y,z) - \frac{1}{2}x^2F^1(y,z), \tag{2.43}$$

for some arbitrary functions $F^2(y,z)$ and $F^3(y,z)$. Thus, equation (2.43) amends equations (2.39) and we obtain

$$\begin{aligned}
\xi^2_z + \xi^3_y &= 0, && (2.44a)\\
xF^1 + \xi^3_z - F^3 &= 0, && (2.44b)\\
xF^1 + \xi^2_y - F^3 &= 0, && (2.44c)\\
x^2F^1_y - 2\left(F^2_y + xF^3_y + \xi^2_x\right) &= 0, && (2.44d)\\
x^2F^1_z - 2\left(F^2_z + xF^3_z + \xi^3_x\right) &= 0, && (2.44e)\\
a_{zz} + a_{yy} + a_{xx} - \frac{1}{2}u\left(\xi^2_{yzz} + \xi^2_{yyy} + \xi^2_{xxy}\right) &= 0, && (2.44f)\\
2F^1(y,z) + x^2\left(F^1_{zz} + F^1_{yy}\right) - \left(F^2_{zz} + xF^3_{zz} + F^2_{yy} + xF^3_{yy} + \xi^2_{xy}\right) &= 0, && (2.44g)\\
\xi^2_{zz} + 2\xi^2_{yy} + \xi^2_{xx} &= 0, && (2.44h)\\
\xi^3_{zz} + \xi^2_{yz} + \xi^3_{yy} + \xi^3_{xx} &= 0, && (2.44i)\\
\xi^3_z - \xi^2_y &= 0, && (2.44j)\\
\xi^2_z + \xi^3_y &= 0, && (2.44k)\\
xF^1 + \xi^2_y - F^3 &= 0, && (2.44l)\\
x^2F^1_y - \left(F^2_y + xF^3_y + \xi^2_x\right) &= 0, && (2.44m)\\
x^2F^1_z - \left(F^2_z + xF^3_z + \xi^3_x\right) &= 0, && (2.44n)\\
vxF^1_{zz} + vxF^1_{yy} + 2\left(d_{zz} + d_{yy} + d_{xx}\right) - v\left(F^3_{zz} + F^3_{yy}\right) &= 0, && (2.44o)\\
4F^1 + x^2\left(F^1_{zz} + F^1_{yy}\right) - \left(F^2_{zz} + xF^3_{zz} + F^2_{yy} + xF^3_{yy}\right) &= 0, && (2.44p)\\
xF^1_y - F^3_y + \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} &= 0, && (2.44q)\\
xF^1_z - F^3_z - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} &= 0. && (2.44r)
\end{aligned}$$

Using (2.44a) and (2.44j) we get

$$\begin{aligned}
\xi^2_{yy} + \xi^2_{zz} &= 0, && (2.45a)\\
\xi^3_{yy} + \xi^3_{zz} &= 0. && (2.45b)
\end{aligned}$$

Substituting (2.45a,b) into (2.44q,r) respectively gives

$$\begin{aligned}
\xi^2 &= F^4(y,z) + xF^5(y,z) + \frac{1}{6}x^3F^1_y - \frac{1}{2}x^2F^3_y, && (2.46a)\\
\xi^3 &= F^6(y,z) + xF^7(y,z) + \frac{1}{6}x^3F^1_y - \frac{1}{2}x^2F^3_y, && (2.46b)
\end{aligned}$$

where F^4, F^5, F^6 and F^7 are arbitrary functions of y and z. Substitution of equations (2.46a,b) into equations (2.44) yields

$$3F^4_z + 3xF^5_z + 3F^6_y + 3xF^7_y + x^3F^1_{yz} - 3x^2F^3_{yz} = 0, \tag{2.47a}$$
$$6xF^1(y,z) + 6F^6_z + 6xF^7_z + x^3F^1_{zz} - 6F^3(y,z) - 3x^2F^3_{zz} = 0, \tag{2.47b}$$
$$6xF^1(y,z) + 6F^4_y + 6xF^5_y + x^3F^1_{yy} - 6F^3(y,z) - 3x^2F^3_{yy} = 0, \tag{2.47c}$$
$$F^5 + F^2_y = 0, \tag{2.47d}$$
$$F^7 + F^2_z = 0, \tag{2.47e}$$
$$\begin{aligned} &u\left(6F^4_{yzz} + 6xF^5_{yzz} + 6xF^1_{yy} + x^3F^1_{yyzz} + 6F^4_{yyy} + 6xF^5_{yyy} + x^3F^1_{yyyy}\right) \\ &-3\left(2uF^3_{yy} + ux^2F^3_{yyzz} + ux^2F^3_{yyyy} + 4a_{zz} + 4a_{yy} + 4a_{xx}\right) = 0, \end{aligned} \tag{2.47f}$$
$$F^1(y,z) + \frac{1}{2}x^2F^1_{zz} - F^2_{zz} - xF^3_{zz} - F^5_y + F^2_{yy} = 0, \tag{2.47g}$$
$$F^3_y + \frac{1}{2}x^2\left(F^3_{yzz} + 2F^3_{yyy}\right) - F^4_{zz} - xF^5_{zz} - xF^1_y + \frac{1}{6}x^3F^1_{yzz} - 2F^4_{yy} - 2xF^5_{yy} - \frac{1}{3}x^3F^1_{yyy} = 0, \tag{2.47h}$$
$$F^3_z + \frac{1}{2}x^2\left(F^3_{zzz} + 2F^3_{yyz}\right) - xF^1_z - F^6_{zz} - xF^7_{zz} - \frac{1}{6}x^3F^1_{zzz} - F^4_{yz} - xF^5_{yz} - F^6_{yy} - xF^7_{yy} - \frac{1}{3}x^3F^1_{yyz} = 0, \tag{2.47i}$$
$$6F^6_z + 6xF^7_z + x^3F^1_{zz} + 3x^2F^3_{yy} - 3x^2F^3_{zz} - 6F^4_y - 6xF^5_y - x^3F^1_{yy} = 0, \tag{2.47j}$$
$$3F^4_z + 3xF^5_z + 3F^6_y + 3xF^7_y + x^3F^1_{yz} - 3x^2F^3_{yz} = 0, \tag{2.47k}$$
$$6xF^1 + 6F^4_y + 6xF^5_y + x^3F^1_{yy} - 6F^3(y,z) - 3x^2F^3_{yy} = 0, \tag{2.47l}$$
$$F^5 + F^2_y = 0, \tag{2.47m}$$
$$F^7 + F^2_z = 0, \tag{2.47n}$$
$$vxF^1_{zz} + vxF^1_{yy} + 2\left(d_{zz} + d_{yy} + d_{xx}\right) - v\left(F^3_{zz} + F^3_{yy}\right) = 0, \tag{2.47o}$$
$$4F^1 + x^2\left(F^1_{zz} + F^1_{yy}\right) - 2\left(F^2_{zz} + xF^3_{zz} + F^2_{yy} + xF^3_{yy}\right) = 0, \tag{2.47p}$$
$$6F^4_{zz} + 6xF^5_{zz} + x^3F^1_{yzz} + 6F^4_{yy} + 6xF^5_{yy} + x^3F^1_{yyy} - 3x^2\left(F^3_{yzz} + F^3_{yyy}\right) = 0, \tag{2.47q}$$
$$6F^6_{zz} + 6xF^7_{zz} + x^3F^1_{zzz} + 6F^6_{yy} + 6xF^7_{yy} + x^3F^1_{yyz} - 3x^2\left(F^3_{zzz} + F^3_{yyz}\right) = 0. \tag{2.47r}$$

From (2.47d,m) and (2.47e,n) we discover that

$$F^5 = -F^2_y, \tag{2.48a}$$
$$F^7 = -F^2_z, \tag{2.48b}$$

respectively. Equations (2.48a,b) satisfy equations (2.47d,m) and (2.47e,n) and the rest of the equations in (2.47) become

$$3F^4_z + 3F^6_y + x^3F^1_{yz} - 3x\left(2F^2_{yz} + xF^3_{yz}\right) = 0, \tag{2.49a}$$
$$6xF^1(y,z) + 6F^6_z + x^3F^1_{zz} - 6F^3(y,z) - 6xF^2_{zz} - 3x^2F^3_{zz} = 0, \tag{2.49b}$$
$$6xF^1(y,z) + 6F^4_y + x^3F^1_{yy} - 6F^3(y,z) - 6xF^2_{yy} - 3x^2F^3_{yy} = 0, \tag{2.49c}$$
$$\begin{aligned} &3\left(2uF^3_{yy} + 2uxF^2_{yyzz} + ux^2F^3_{yyzz} + 2uxF^2_{yyyy} + ux^2F^3_{yyyy} + 4a_{zz} + 4a_{yy} + 4a_{xx}\right) \\ &-u\left(6F^4_{yzz} + 6xF^1_{yy} + x^3F^1_{yyzz} + 6F^4_{yyy} + x^3F^1_{yyyy}\right) = 0, \end{aligned} \tag{2.49d}$$
$$F^1(y,z) + \frac{1}{2}x^2F^1_{zz} - F^2_{zz} - xF^3_{zz} = 0, \tag{2.49e}$$
$$-F^3_y - \frac{1}{2}x\left(2F^2_{yzz} + xF^3_{yzz} + 4F^2_{yyy} + 2xF^3_{yyy}\right) + F^4_{zz} + xF^1_y + \frac{1}{6}x^3F^1_{yzz} + 2F^4_{yy} + \frac{1}{3}x^3F^1_{yyy} = 0, \tag{2.49f}$$

$$-F^3_z - \frac{1}{2}x\left(2F^2_{zzz} + xF^3_{zzz} + 4F^2_{yyz} + 2xF^3_{yyz}\right) + xF^1_z + F^6_{zz} + \frac{1}{6}x^3F^1_{zzz} + F^4_{yz} + F^6_{yy} + \frac{1}{3}x^3F^1_{yyz} = 0, \tag{2.49g}$$

$$6F^6_z + x^3F^1_{zz} + 6xF^2_{yy} + 3x^2F^3_{yy} - 6xF^2_{zz} - 3x^2F^3_{zz} - 6F^4_y - x^3F^1_{yy} = 0, \tag{2.49h}$$

$$3F^4_z + 3F^6_y + x^3F^1_{yz} - 3x\left(2F^2_{yz} + xF^3_{yz}\right) = 0, \tag{2.49i}$$

$$6xF^1(y,z) + 6F^4_y + x^3F^1_{yy} - 6F^3 - 6xF^2_{yy} - 3x^2F^3_{yy} = 0, \tag{2.49j}$$

$$vxF^1_{zz} + vxF^1_{yy} + 2\left(d_{zz} + d_{yy} + d_{xx}\right) - v\left(F^3_{zz} + F^3_{yy}\right) = 0, \tag{2.49k}$$

$$4F^1 + x^2\left(F^1_{zz} + F^1_{yy}\right) - 2\left(F^2_{zz} + xF^3_{zz} + F^2_{yy} + xF^3_{yy}\right) = 0, \tag{2.49l}$$

$$6F^4_{zz} + x^3F^1_{yzz} + 6F^4_{yy} + x^3F^1_{yyy} - 3x\left(2F^2_{yzz} + xF^3_{yzz} + 2F^2_{yyy} + xF^3_{yyy}\right) = 0. \tag{2.49m}$$

The functions F^1, F^2, F^3 and F^6 do not depend on x, hence from (2.49b) it can be concluded that

$$F^1 = e^1(y) + ze^2(y), \tag{2.50a}$$

$$F^2 = e^3(y) + ze^4(y), \tag{2.50b}$$

where e^1, e^2, e^3 and e^4 are arbitrary functions of y. Updating equations (2.49) using (2.50a,b) gives

$$x^3e^2_y + 3\left(F^4_z + F^6_y\right) - 3x\left(xe^4_y + 2F^2_{yz}\right) = 0, \tag{2.51a}$$

$$xe^1 + xze^2 + F^6_z - e^3 - ze^4 - xF^2_{zz} = 0, \tag{2.51b}$$

$$-3\left(2e^3 + 2ze^4x\left(xe^3_{yy} + xze^4_{yy} + 2F^2_{yy}\right)\right) + 6xe^1 + 6xze^2 + x^3e^1_{yy} + x^3ze^2_{yy} + 6F^4_y = 0, \tag{2.51c}$$

$$\begin{aligned} &u\left(6xe^1_{yy} + 6xze^2_{yy} + x^3e^1_{yyyy} + x^3ze^2_{yyyy} + 6F^4_{yzz} + 6F^4_{yyy}\right) \\ &-3\left(2ue^3_{yy} + 2uze^4_{yy} + ux^2e^3_{yyyy} + ux^2ze^4_{yyyy} + 2uxF^2_{yyzz} + 2uxF^2_{yyyy} + 4a_{zz} + 4a_{yy} + 4a_{xx}\right) = 0, \end{aligned} \tag{2.51d}$$

$$e^1 + ze^2 - F^2_{zz} = 0, \tag{2.51e}$$

$$-e^3_y - ze^4_y - x\left(xe^3_{yyy} + xze^4_{yyy} + F^2_{yzz} + 2F^2_{yyy}\right) + xe^1_y + xze^2_y + \frac{1}{3}x^3e^1_{yyy} + \frac{1}{3}x^3ze^2_{yyy} + F^4_{zz} + 2F^4_{yy} = 0, \tag{2.51f}$$

$$-e^4 - x\left(xe^4_{yy} + F^2_{zzz} + 2F^2_{yyz}\right) + xe^2 + \frac{1}{3}x^3e^2_{yy} + F^6_{zz} + F^4_{yz} + F^6_{yy} = 0, \tag{2.51g}$$

$$x^3\left(e^1_{yy} + ze^2_{yy}\right) - 3\left(x^2e^3_{yy} + x^2ze^4_{yy} + 2\left(F^6_z - xF^2_{zz} - F^4_y + xF^2_{yy}\right)\right) = 0, \tag{2.51h}$$

$$x^3e^2_y + 3\left(F^4_z + F^6_y\right) - 3x\left(xe^4_y + 2F^2_{yz}\right) = 0, \tag{2.51i}$$

$$6xe^1 + 6xze^2 + x^3e^1_{yy} + x^3ze^2_{yy} + 6F^4_y - 3\left(2e^3 + 2ze^4 + x\left(xe^3_{yy} + xze^4_{yy} + 2F^2_{yy}\right)\right) = 0, \tag{2.51j}$$

$$vxe^1_{yy} + vxze^2_{yy} + 2\left(d_{zz} + d_{yy} + d_{xx}\right) - v\left(e^3_{yy} + ze^4_{yy}\right) = 0, \tag{2.51k}$$

$$4e^1 + 4ze^2 + x^2\left(e^1_{yy} + ze^2_{yy}\right) - 2\left(xe^3_{yy} + xze^4_{yy} + F^2_{zz} + F^2_{yy}\right) = 0, \tag{2.51l}$$

$$x^3\left(e^1_{yyy} + ze^2_{yyy}\right) - 3\left(x^2e^3_{yyy} + x^2ze^4_{yyy} - 2\left(F^4_{zz} - xF^2_{yzz} + F^4_{yy} - xF^2_{yyy}\right)\right) = 0, \tag{2.51m}$$

$$-\frac{1}{2}x\left(xe^4_{yy} + 2\left(F^2_{zzz} + F^2_{yyz}\right)\right) + \frac{1}{6}x^3e^2_{yy} + F^6_{zz} + F^6_{yy} = 0. \tag{2.51n}$$

Upon separating (2.51a) by powers of x we find that $e^2 = K_5$ and $e^4 = K_6$ where K_5 and K_6 where K_5 and K_6 are arbitrary constants. Further differentiating (2.51k) with respect to v and then separating the result by powers of x and then by z we find that e^1 and e^3 are both linear in y, i.e.,

$$e^1 = K_7 + yK_8, \tag{2.52a}$$

$$e^3 = K_9 + yK_{10}. \tag{2.52b}$$

Updating equations (2.51) taking into account e^1, e^2, e^3 and e^4 we get

$$F^4_z + F^6_y - 2xF^2_{yz} = 0, \tag{2.53a}$$
$$-x(zK_5 + K_7 + yK_8) - F^6_z + zK_6 + K_9 + yK_{10,} + xF^2_{zz} = 0, \tag{2.53b}$$
$$-x(zK_5 + K_7 + yK_8) - F^4_y + zK_6 + K_9 + yK_{10,} + xF^2_{yy} = 0, \tag{2.53c}$$
$$-u\left(F^4_{yzz} + F^4_{yyy}\right) + uxF^2_{yyzz} + uxF^2_{yyyy} + 2\left(a_{zz} + a_{yy} + a_{xx}\right) = 0, \tag{2.53d}$$
$$zK_5 + K_7 + yK_8 - F^2_{zz} = 0, \tag{2.53e}$$
$$xK_8 + F^4_{zz} + 2F^4_{yy} - K_{10,} - xF^2_{yzz} - 2xF^2_{yyy} = 0, \tag{2.53f}$$
$$xK_5 + F^6_{zz} + F^4_{yz} + F^6_{yy} - K_6 - xF^2_{zzz} - 2xF^2_{yyz} = 0, \tag{2.53g}$$
$$F^6_z + xF^2_{yy} - xF^2_{zz} - F^4_y = 0, \tag{2.53h}$$
$$F^4_z + F^6_y - 2xF^2_{yz} = 0, \tag{2.53i}$$
$$-x(zK_5 + K_7 + yK_8) - F^4_y + zK_6 + K_9 + yK_{10,} + xF^2_{yy} = 0, \tag{2.53j}$$
$$d_{zz} + d_{yy} + d_{xx} = 0, \tag{2.53k}$$
$$-2(zK_5 + K_7 + yK_8) + F^2_{zz} + F^2_{yy} = 0, \tag{2.53l}$$
$$-x\left(F^2_{yzz} + F^2_{yyy}\right) + F^4_{zz} + F^4_{yy} = 0, \tag{2.53m}$$
$$-x\left(F^2_{zzz} + F^2_{yyz}\right) + F^6_{zz} + F^6_{yy} = 0. \tag{2.53n}$$

Differentiating (2.53j) with respect to x gives

$$F^2_{yy} = zK_5 + K_7 + yK_8, \tag{2.54}$$

which implies that

$$F^2 = \frac{1}{2}y^2K_7 + \frac{1}{6}y^3K_8 + e^5(z) + ye^6(z), \tag{2.55}$$

where e^5 and e^6 are arbitrary functions of z. Thus, (2.55) updates equations (2.53) as follows

$$-2xe^6_z + F^4_z + F^6_y = 0, \tag{2.56a}$$
$$-x(zK^5 + K_7 + yK_8) - F^6_z + zK_6 + K_9 + yK_{10,} + xe^5_{zz} + xye^6_{zz} = 0, \tag{2.56b}$$
$$xzK^5 + F^4_y - zK_6 - K_9 - yK_{10,} = 0, \tag{2.56c}$$
$$a_{zz} + a_{yy} + a_{xx} - \frac{1}{2}u\left(F^4_{yzz} + F^4_{yyy}\right) = 0, \tag{2.56d}$$
$$zK^5 + K_7 + yK_8 - e^5_{zz} - ye^6_{zz} = 0, \tag{2.56e}$$
$$xK_8 + K_{10,} + xe^6_{zz} - F^4_{zz} - 2F^4_{yy} = 0, \tag{2.56f}$$
$$xK^5 + F^6_{zz} + F^4_{yz} + F^6_{yy} - K_6 - xe^5_{zzz} - xye^6_{zzz} = 0, \tag{2.56g}$$
$$xK_7 + xyK_8 + F^6_z - xe^5_{zz} - xye^6_{zz} - F^4_y = 0, \tag{2.56h}$$
$$2xe^6_z - F^4_z - F^6_y = 0, \tag{2.56i}$$
$$xzK^5 + F^4_y - zK_6 - K_9 - yK_{10,} = 0, \tag{2.56j}$$

$$d_{zz}+d_{yy}+d_{xx}=0, \tag{2.56k}$$
$$2zK^5+K_7+yK_8-e^5_{zz}-ye^6_{zz}=0, \tag{2.56l}$$
$$-x\left(K_8+e^6_{zz}\right)+F^4_{zz}+F^4_{yy}=0, \tag{2.56m}$$
$$-x\left(e^5_{zzz}+ye^6_{zzz}\right)+F^6_{zz}+F^6_{yy}=0. \tag{2.56n}$$

Differentiating (2.56a) with respect to x gives $e^6_z=0$, which implies that $e^6=K_{11}$, for some arbitrary constant K_{11}. Again, differentiating (2.56j) with respect to x and (2.56e) with respect to y we get $K_5=K_8=0$. Thus, from (2.56l) we get

$$e^5=K_{12}+zK_{13}+\frac{1}{2}z^2K_7. \tag{2.57}$$

The remaining equations in (2.56) become

$$F^4_z+F^6_y=0, \tag{2.58a}$$
$$zK_6+K_9+yK_{10}-F^6_z=0, \tag{2.58b}$$
$$zK_6+K_9+yK_{10}-F^4_y=0, \tag{2.58c}$$
$$a_{zz}+a_{yy}+a_{xx}-\frac{1}{2}u\left(F^4_{yzz}+F^4_{yyy}\right)=0, \tag{2.58d}$$
$$-K_{10}+F^4_{zz}+2F^4_{yy}=0, \tag{2.58e}$$
$$-K_6+F^6_{zz}+F^4_{yz}+F^6_{yy}=0, \tag{2.58f}$$
$$F^6_z-F^4_y=0, \tag{2.58g}$$
$$F^4_z+F^6_y=0, \tag{2.58h}$$
$$zK_6+K_9+yK_{10,}-F^4_y=0, \tag{2.58i}$$
$$d_{zz}+d_{yy}+d_{xx}=0, \tag{2.58j}$$
$$F^4_{zz}+F^4_{yy}=0, \tag{2.58k}$$
$$F^6_{zz}+F^6_{yy}=0. \tag{2.58l}$$

From (2.58b,c) we respectively get

$$F^6=\frac{1}{2}z^2K_6+zK_9+yzK_{10}+e^7(y), \tag{2.59a}$$
$$F^4=yzK_6+yK_9+\frac{1}{2}y^2K_{10}+e^8(z), \tag{2.59b}$$

where e^7 and e^8 are arbitrary functions of y and z respectively. Substituting (2.59a,b) into (2.58) gives

$$a_{zz}+a_{yy}+a_{xx}=0, \tag{2.60a}$$
$$K_{10}+e^8_{zz}=0, \tag{2.60b}$$
$$K_6+e^7_{yy}=0, \tag{2.60c}$$
$$yK_6+zK_{10}+e^7_y+e^8_z=0, \tag{2.60d}$$
$$d_{zz}+d_{yy}+d_{xx}=0, \tag{2.60e}$$

$$K_{10} + e^8_{zz} = 0, \tag{2.60f}$$

$$K_6 + e^7_{yy} = 0. \tag{2.60g}$$

Solving fo e^8 from (2.60f) we get

$$e^8 = K_{14} + zK_{15} - \frac{1}{2}z^2K_{10}, \tag{2.61}$$

where K_{14} and K_{15} are arbitrary constants. Substituting (2.61) in to (2.60d) gives

$$yK_6 + K_{15} + e^7_y = 0. \tag{2.62}$$

Finally solving for e^7 from (2.62) yields

$$e^7 = K_{15} - \frac{1}{2}y^2K_6 - yK_{15}. \tag{2.63}$$

Substituting (2.61) and (2.63) into equations (2.60) we obtain

$$\begin{aligned} a_{xx} + a_{yy} + a_{zz} &= 0, \\ d_{xx} + d_{yy} + d_{zz} &= 0, \end{aligned} \tag{2.64}$$

which is system (2.34). Thus we have the solutions

$$\xi^1 = (-x^2 + y^2 + z^2)K_7 + 2x(zK_6 + K_9 + yK_{10}) + 2yK_{11} + 2K_{12} + 2zK_{13}, \tag{2.65a}$$

$$\xi^2 = 2yzK_6 + 2yK_9 + (-x^2 + y^2 - z^2)K_{10} - 2x(yK_7 + K_{11})2K_{14} + 2zK_{15}, \tag{2.65b}$$

$$\xi^3 = (-x^2 - y^2 + z^2)K_6 + 2zK_9 + 2yzK_{10} - 2x(zK_7 + K_{13}) - 2yK_{15} + 2K_{16}, \tag{2.65c}$$

$$\eta^1 = 2vK_1 + u(2K_3 - zK_6 + xK_7 - K_9 - yK_{10}) + 2a(x,y,z), \tag{2.65d}$$

$$\eta^2 = 2uK_2 + v(2K_4 - zK_6 + xK_7 - K_9 - yK_{10}) + 2d(x,y,z), \tag{2.65e}$$

and hence the symmetries are

$$X_1 = \partial_x, \tag{2.66a}$$

$$X_2 = \partial_y, \tag{2.66b}$$

$$X_3 = \partial_z, \tag{2.66c}$$

$$X_4 = y\partial_x - x\partial_y, \tag{2.66d}$$

$$X_5 = z\partial_x - x\partial_z, \tag{2.66e}$$

$$X_6 = z\partial_y - y\partial_z, \tag{2.66f}$$

$$X_7 = 2x\partial_x + 2y\partial_y + 2z\partial_z - u\partial_u - v\partial_v, \tag{2.66g}$$

$$X_8 = 2xy\partial_x + (-x^2 + y^2 - z^2)\partial_y + 2yz\partial_z - yu\partial_u - yv\partial_v, \tag{2.66h}$$

$$X_9 = 2xz\partial_x + 2yz\partial_y + (-x^2 - y^2 + z^2)\partial_z - zu\partial_u - zv\partial_v, \tag{2.66i}$$

$$X_{10} = (-x^2 + y^2 + z^2)\partial_x - xy\partial_y - 2xz\partial_z + xu\partial_u + xv\partial_v, \tag{2.66j}$$

$$X_{11} = u\partial_u, \tag{2.66k}$$

$$X_{12} = v\partial_u, \tag{2.66l}$$

$$X_{13} = u\partial_v, \tag{2.66m}$$

$$X_{14} = v\partial_v, \tag{2.66n}$$

$$X_a = a(x,y,z)\partial_u, \tag{2.66o}$$

$$X_d = d(x,y,z)\partial_v, \tag{2.66p}$$

where $a(x,y,z)$ and $d(x,y,z)$ satisfy system (2.34) or (2.64).

2.3 System 4

Consider system (1.6d) in expanded form

$$u_{xx} + u_{yy} + u_{zz} = u, \tag{2.67a}$$

$$v_{xx} + v_{yy} + v_{zz} = v. \tag{2.67b}$$

Likewise, the symmetry criterion for system (2.67) yields the following determining equations

$$\xi^1_v = 0, \tag{2.68a}$$

$$\xi^2_v = 0, \tag{2.68b}$$

$$\xi^3_v = 0, \tag{2.68c}$$

$$\eta^1_{vv} = 0, \tag{2.68d}$$

$$\eta^2_{vv} = 0, \tag{2.68e}$$

$$\xi^1_u = 0, \tag{2.68f}$$

$$\xi^2_u = 0, \tag{2.68g}$$

$$\xi^3_u = 0, \tag{2.68h}$$

$$\eta^1_{uv} = 0, \tag{2.68i}$$

$$\eta^2_{uv} = 0, \tag{2.68j}$$

$$\eta^1_{uu} = 0, \tag{2.68k}$$

$$\eta^2_{uu} = 0, \tag{2.68l}$$

$$\eta^1_{zv} = 0, \tag{2.68m}$$

$$\eta^2_{zu} = 0, \tag{2.68n}$$

$$\eta^1_{yv} = 0, \tag{2.68o}$$

$$\eta^2_{yu} = 0, \tag{2.68p}$$

$$\eta^1_{xv} = 0, \tag{2.68q}$$

$$\eta^2_{xu} = 0, \tag{2.68r}$$

$$\xi^2_z + \xi^3_y = 0, \tag{2.68s}$$

$$\xi^3_z - \xi^1_x = 0, \tag{2.68t}$$

$$\xi^2_y - \xi^1_x = 0, \tag{2.68u}$$

$$\xi^1_y + \xi^2_x = 0, \tag{2.68v}$$

$$\xi^1_z + \xi^3_x = 0, \tag{2.68w}$$

$$v\eta^1_v - \eta^1 + u\eta^1_u + \eta^1_{zz} + \eta^1_{yy} - 2u\xi^1_x + \eta^1_{xx} = 0, \tag{2.68x}$$

$$2\eta^1_{xu} - \xi^1_{zz} - \xi^1_{yy} - \xi^1_{xx} = 0, \tag{2.68y}$$

$$2\eta^1_{yu} - \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} = 0, \tag{2.68z}$$

$$2\eta^1_{zu} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0, \tag{2.68aa}$$

$$\xi^3_z - \xi^2_y = 0, \tag{2.68ab}$$

$$\xi^2_z + \xi^3_y = 0, \tag{2.68ac}$$

$$\xi^2_y - \xi^1_x = 0, \tag{2.68ad}$$

$$\xi^1_y + \xi^2_x = 0, \tag{2.68ae}$$

$$\xi^1_z + \xi^3_x = 0, \tag{2.68af}$$
$$v\eta^2_v - \eta^2 + u\eta^2_u + \eta^2_{zz} - 2v\xi^2_y + \eta^2_{yy} + \eta^2_{xx} = 0, \tag{2.68ag}$$
$$2\eta^2_{xv} - \xi^1_{zz} - \xi^1_{yy} - \xi^1_{xx} = 0, \tag{2.68ah}$$
$$2\eta^2_{yv} - \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} = 0, \tag{2.68ai}$$
$$2\eta^2_{zv} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0. \tag{2.68aj}$$

Equations (2.68a,r) solve to

$$\xi^1 = \xi^1(x,y,z), \tag{2.69a}$$
$$\xi^2 = \xi^2(x,y,z), \tag{2.69b}$$
$$\xi^3 = \xi^3(x,y,z), \tag{2.69c}$$
$$\eta^1 = a(x,y,z) + \frac{u}{2}\left(2K_3 - \xi^2_y\right) + vK_1, \tag{2.69d}$$
$$\eta^2 = d(x,y,z) + uK_2 + \frac{v}{2}\left(2K_4 - \xi^2_y\right), \tag{2.69e}$$

where a and d are arbitrary functions of x, y, z and K_1, K_2, K_3 and K_4 are arbitrary constants. Due to equations (2.69), equations (2.68s) to (2.68aj) become

$$\xi^2_z + \xi^3_y = 0, \tag{2.70a}$$
$$\xi^3_z - \xi^1_x = 0, \tag{2.70b}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.70c}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.70d}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.70e}$$
$$a_{zz} + a_{yy} + a_{xx} - \frac{1}{2}\left(2a + u\left(\xi^2_{yzz} + \xi^2_{yyy} + 4\xi^1_x + \xi^2_{xxy}\right)\right) = 0, \tag{2.70f}$$
$$\xi^1_{zz} + \xi^1_{yy} + \xi^2_{xy} + \xi^1_{xx} = 0, \tag{2.70g}$$
$$\xi^2_{zz} + 2\xi^2_{yy} + \xi^2_{xx} = 0, \tag{2.70h}$$
$$\xi^3_{zz} + \xi^2_{yz} + \xi^3_{yy} + \xi^3_{xx} = 0, \tag{2.70i}$$
$$\xi^3_z - \xi^2_y = 0, \tag{2.70j}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.70k}$$
$$\xi^2_y - \xi^1_x = 0, \tag{2.70l}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.70m}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.70n}$$
$$d_{zz} + d_{yy} + d_{xx} - \frac{1}{2}\left(2d + v\left(4\xi^2_y + \xi^2_{yzz} + \xi^2_{yyy} + \xi^2_{xxy}\right)\right) = 0, \tag{2.70o}$$
$$\xi^1_{zz} + \xi^1_{yy} + \xi^2_{xy} + \xi^1_{xx} = 0, \tag{2.70p}$$
$$\xi^2_{zz} + 2\xi^2_{yy} + \xi^2_{xx} = 0, \tag{2.70q}$$
$$\xi^3_{zz} + \xi^2_{yz} + \xi^3_{yy} + \xi^3_{xx} = 0. \tag{2.70r}$$

Differentiating (2.70o) with respect to v we get

$$4\xi^2_y + \xi^2_{xxy} + \xi^2_{yyy} + \xi^2_{yzz} = 0. \tag{2.71}$$

Integrating (2.71) with respect to y yields

$$4\xi^2 + \xi^2_{xx} + \xi^2_{yy} + \xi^2_{zz} = G^1(x,z). \tag{2.72}$$

where $G^1(x,z)$ is an arbitrary function. Substituting (2.70q) into (2.72) yields the differential equation

$$4\xi^2 - \xi^2_{yy} = G^1(x,z), \tag{2.73}$$

which solves for ξ^2 to

$$\xi^2 = e^{2y}G^2(x,z) + e^{-2y}G^3(x,z) + \frac{1}{4}G^1(x,z). \tag{2.74}$$

Substitution of (2.74) into equations (2.70) ultimately gives $G^2(x,z) = G^3(x,z) = 0$. Thus

$$\xi^2 = \frac{1}{4}G^1(x,z), \tag{2.75}$$

which is similar to

$$\xi^2 = \xi^2(x,z). \tag{2.76}$$

Using (2.76) in (2.70c) yields

$$\xi^1 = \xi^1(y,z), \tag{2.77}$$

and (2.77) in (2.70b) yields

$$\xi^3 = \xi^3(x,y). \tag{2.78}$$

Updating equations (2.70) with (2.76), (2.77) and (2.78) gives the following equations

$$\xi^2_z + \xi^3_y = 0, \tag{2.79a}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.79b}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.79c}$$
$$-a + a_{zz} + a_{yy} + a_{xx} = 0, \tag{2.79d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.79e}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.79f}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0, \tag{2.79g}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.79h}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.79i}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.79j}$$
$$-d + d_{zz} + d_{yy} + d_{xx} = 0, \tag{2.79k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.79l}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.79m}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0. \tag{2.79n}$$

From (2.79b) and (2.79c) we get

$$\xi^2 = -x\xi^1_y + c^1(z), \tag{2.80a}$$
$$\xi^3 = -x\xi^1_z + c^2(y), \tag{2.80b}$$

where c^1 and c^2 are functions of z and y respectively. Substituting equations (2.80a) and (2.80b) into equations (2.79) we get the following equations

$$c^1_z + c^2_y - 2x\xi^1_{yz} = 0, \tag{2.81a}$$
$$x\xi^1_{zz} = 0, \tag{2.81b}$$
$$x\xi^1_{yy} = 0, \tag{2.81c}$$
$$-a + a_{zz} + a_{yy} + a_{xx} \tag{2.81d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.81e}$$
$$-c^1_{zz} + x\left(\xi^1_{yzz} + \xi^1_{yyy}\right) = 0, \tag{2.81f}$$
$$-c^2_{yy} + x\left(\xi^1_{zzz} + \xi^1_{yyz}\right) = 0, \tag{2.81g}$$
$$x\xi^1_{zz} - x\xi^1_{yy} = 0, \tag{2.81h}$$
$$c^1_z + c^2_y - 2x\xi^1_{yz} = 0, \tag{2.81i}$$
$$x\xi^1_{yy} = 0, \tag{2.81j}$$
$$-d + 2vx\xi^1_{yy} + d_{zz} + d_{yy} + d_{xx} = 0, \tag{2.81k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.81l}$$
$$-c^1_{zz} + x\left(\xi^1_{yzz} + \xi^1_{yyy}\right) = 0, \tag{2.81m}$$
$$-c^2_{yy} + x\left(\xi^1_{zzz} + \xi^1_{yyz}\right) = 0. \tag{2.81n}$$

Equation (2.81c) implies that ξ^1 is linear in y,

$$\xi^1 = c^3 + ye^4. \tag{2.82}$$

Using (2.82) into (2.81i) and separating with powers of x gives

$$c^4 = K_5, \tag{2.83}$$

where K_5 is an arbitrary constant. Further substituting (2.82) and (2.83) into (2.81b) we find that e^3 is linear in z, i.e

$$e^3 = K_6 + zK_7. \tag{2.84}$$

Plugging (2.83) and (2.84) into equations (2.81) yields

$$-a + a_{zz} + a_{yy} + a_{xx} = 0, \tag{2.85a}$$
$$c^2_{yy} = 0, \tag{2.85b}$$
$$K_9 + c^2_y = 0, \tag{2.85c}$$
$$-d + d_{zz} + d_{yy} + d_{xx} = 0. \tag{2.85d}$$

Solving for c^2 from (2.85c) we get

$$c^2 = K_{10} - yK_9. \tag{2.86}$$

Finally the following system remains

$$a_{zz} + a_{yy} + a_{xx} = a, \tag{2.87a}$$
$$d_{zz} + d_{yy} + d_{xx} = d, \tag{2.87b}$$

which is system (2.67). We therefore have the following solutions

$$\xi^1 = yK_5 + K_6 + zK_7, \tag{2.88a}$$
$$\xi^2 = -xK_5 + K_8 + zK_9, \tag{2.88b}$$
$$\xi^3 = -xK_7 - yK_9 + K_{10}, \tag{2.88c}$$
$$\eta^1 = uK_3 + vK_1 + a(x, y, z), \tag{2.88d}$$
$$\eta^2 = uK_2 + vK_4 + d(x, y, z). \tag{2.88e}$$

Hence we obtain the following symmetries

$$X_1 = \partial_x, \tag{2.89a}$$
$$X_2 = \partial_y, \tag{2.89b}$$
$$X_3 = \partial_z. \tag{2.89c}$$
$$X_4 = y\partial_x - x\partial_y, \tag{2.89d}$$
$$X_5 = z\partial_x - x\partial_z, \tag{2.89e}$$
$$X_6 = z\partial_y - y\partial_z, \tag{2.89f}$$
$$X_7 = u\partial_u, \tag{2.89g}$$
$$X_8 = v\partial_u, \tag{2.89h}$$
$$X_9 = u\partial_v, \tag{2.89i}$$
$$X_{10} = v\partial_v, \tag{2.89j}$$
$$X_a = a(x, y, z)\partial_u, \tag{2.89k}$$
$$X_b = b(x, y, z)\partial_v, \tag{2.89l}$$

where $a(x, y, z)$ and $b(x, y, z)$ solve system (2.67) or (2.87).

System (1.6e) in expanded form reads

$$u_{xx} + u_{yy} + u_{zz} = -u, \tag{2.90a}$$
$$v_{xx} + v_{yy} + v_{zz} = -v, \tag{2.90b}$$

which is system 5, it possesses symmetries (2.89).

2.4 System 6

For the system of PDEs

$$u_{xx}+u_{yy}+u_{zz}=\alpha u+v, \qquad \alpha\geq 0 \tag{2.91a}$$

$$v_{xx}+v_{yy}+v_{zz}=v, \tag{2.91b}$$

which is the expanded form of (1.6f), the symmetry criterion for system (2.91) yields the following determining equations

$$\xi^1_v=0, \tag{2.92a}$$
$$\xi^2_v=0, \tag{2.92b}$$
$$\xi^3_v=0, \tag{2.92c}$$
$$\eta^1_{vv}=0, \tag{2.92d}$$
$$\eta^2_{vv}=0, \tag{2.92e}$$
$$\xi^1_u=0, \tag{2.92f}$$
$$\xi^2_u=0, \tag{2.92g}$$
$$\xi^3_u=0, \tag{2.92h}$$
$$\eta^1_{uv}=0, \tag{2.92i}$$
$$\eta^2_{uv}=0, \tag{2.92j}$$
$$\eta^1_{uu}=0, \tag{2.92k}$$
$$\eta^2_{uu}=0, \tag{2.92l}$$
$$\eta^1_{zv}=0, \tag{2.92m}$$
$$\eta^2_{zu}=0, \tag{2.92n}$$
$$\eta^1_{yv}=0, \tag{2.92o}$$
$$\eta^2_{yu}=0, \tag{2.92p}$$
$$\eta^1_{xv}=0, \tag{2.92q}$$
$$\eta^2_{xu}=0, \tag{2.92r}$$
$$\xi^2_z+\xi^3_y=0, \tag{2.92s}$$
$$\xi^3_z-\xi^1_x=0, \tag{2.92t}$$
$$\xi^2_y-\xi^1_x=0, \tag{2.92u}$$
$$\xi^1_y\xi^2_x=0, \tag{2.92v}$$
$$\xi^1_z+\xi^3_x=0, \tag{2.92w}$$
$$-\alpha\eta^1-\eta^2+v\eta^1_v+(v+u\alpha)\eta^1_u+\eta^1_{zz}+\eta^1_{yy}+(-2v-2u\alpha)\xi^1_x+\eta^1_{xx}=0, \tag{2.92x}$$
$$2\eta^1_{xu}-\xi^1_{zz}-\xi^1_{yy}-\xi^1_{xx}=0, \tag{2.92y}$$
$$2\eta^1_{yu}-\xi^2_{zz}-\xi^2_{yy}-\xi^2_{xx}=0, \tag{2.92z}$$
$$2\eta^1_{zu}-\xi^3_{zz}-\xi^3_{yy}-\xi^3_{xx}=0, \tag{2.92aa}$$
$$\xi^3_z-\xi^2_y=0, \tag{2.92ab}$$
$$\xi^2_z\xi^3_y=0, \tag{2.92ac}$$

$$\xi^2_y - \xi^1_x = 0, \qquad (2.92ad)$$
$$\xi^1_y + \xi^2_x = 0, \qquad (2.92ae)$$
$$\xi^1_z + \xi^3_x = 0, \qquad (2.92af)$$
$$\eta^2_{xx} - \eta^2 + v\eta^2_v + (v + u\alpha)\eta^2_u + \eta^2_{zz} - 2v\xi^2_y + \eta^2_{yy} = 0, \qquad (2.92ag)$$
$$2\eta^2_{xv} - \xi^1_{zz} - \xi^1_{yy} - \xi^1_{xx} = 0, \qquad (2.92ah)$$
$$2\eta^2_{yv} - \xi^2_{zz} - \xi^2_{yy} - \xi^2_{xx} = 0, \qquad (2.92ai)$$
$$2\eta^2_{zv} - \xi^3_{zz} - \xi^3_{yy} - \xi^3_{xx} = 0. \qquad (2.92aj)$$

Solving equations (2.92a) to (2.92r) give the following

$$\xi^1 = \xi^1(x, y, z), \qquad (2.93a)$$
$$\xi^2 = \xi^2(x, y, z), \qquad (2.93b)$$
$$\xi^3 = \xi^3(x, y, z), \qquad (2.93c)$$
$$\eta^1 = a(x, y, z) + \frac{u}{2}\left(2K_3 - \xi^2_y\right) + vK_1, \qquad (2.93d)$$
$$\eta^2 = d(x, y, z) + uK_2 + \frac{v}{2}\left(2K_4 - \xi^2_y\right), \qquad (2.93e)$$

where a and d are arbitrary functions and K_1, K_2, K_3, K_4 are arbitrary constants. Equations (2.93) amend (2.92s) to (293aj) thus we have

$$\xi^2_z + \xi^3_y = 0, \qquad (2.94a)$$
$$\xi^3_z - \xi^1_x = 0, \qquad (2.94b)$$
$$\xi^2_y - \xi^1_x = 0, \qquad (2.94c)$$
$$\xi^1_y + \xi^2_x = 0, \qquad (2.94d)$$
$$\xi^1_z + \xi^3_x = 0, \qquad (2.94e)$$
$$-vK_1 - vK_3 - a_{zz} - a_{yy} - a_{xx} + \frac{1}{2}(2v\alpha K_1 + 2uK_2 + 2vK_4) + \frac{1}{2}\left(2\alpha a + 2d + v\xi^2_y + u\xi^2_{yzz} + u\xi^2_{yyy} + 3v\xi^1_x + 4u\alpha\xi^1_x + u\xi^2_{xxy}\right) = 0, \qquad (2.94f)$$
$$\xi^1_{zz} + \xi^1_{yy} + \xi^2_{xy} + \xi^1_{xx} = 0, \qquad (2.94g)$$
$$\xi^2_{zz} + 2\xi^2_{yy} + \xi^2_{xx} = 0, \qquad (2.94h)$$
$$\xi^3_{zz} + \xi^2_{yz} + \xi^3_{yy} + \xi^3_{xx} = 0, \qquad (2.94i)$$
$$\xi^3_z - \xi^2_y = 0, \qquad (2.94j)$$
$$\xi^2_z + \xi^3_y = 0, \qquad (2.94k)$$
$$\xi^2_y - \xi^1_x = 0, \qquad (2.94l)$$
$$\xi^1_y + \xi^2_x = 0, \qquad (2.94m)$$
$$\xi^1_z + \xi^3_x = 0, \qquad (2.94n)$$
$$-vK_2 - u\alpha K_2 - d_{zz} - d_{yy} - d_{xx} + \frac{1}{2}\left(2uK_2 + 2d + 4v\xi^2_y + v\xi^1_{xzz} + v\xi^1_{xyy} + v\xi^1_{xxx}\right) = 0, \qquad (2.94o)$$
$$\xi^1_{zz} + \xi^1_{yy} + 2\xi^1_{xx} = 0, \qquad (2.94p)$$

$$\xi^2_{zz} + \xi^2_{yy} + \xi^1_{xy} + \xi^2_{xx} = 0, \tag{2.94q}$$
$$\xi^3_{zz} + \xi^3_{yy} + \xi^1_{xz} + \xi^3_{xx} = 0. \tag{2.94r}$$

Differentiating (2.94f,o) with respect to u and v and using (2.94d,l) we find that $\xi^1_x = 0$, i.e

$$\xi^1 = \xi^1(y,z). \tag{2.95}$$

When used together with (2.94b,c), (2.95) yields

$$\xi^2 = \xi^2(x,z), \tag{2.96a}$$
$$\xi^3 = \xi^3(y,z). \tag{2.96b}$$

Equations (2.95) and (2.96a,b) satisfy equations (2.94f,o,d,l) and the rest of equations (2.94) become

$$\xi^2_z + \xi^3_y = 0, \tag{2.97a}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.97b}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.97c}$$
$$v\alpha K_1 + uK_2 + vK_4 + \alpha a + d - vK_1 - vK_3 - a_{zz} - a_{yy} - a_{xx} = 0, \tag{2.97d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.97e}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.97f}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0, \tag{2.97g}$$
$$\xi^2_z + \xi^3_y = 0, \tag{2.97h}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.97i}$$
$$\xi^1_z + \xi^3_x = 0, \tag{2.97j}$$
$$vK_2 + u\alpha K_2 + d_{zz} + d_{yy} + d_{xx} - uK_2 - d(x,y,z) = 0, \tag{2.97k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.97l}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.97m}$$
$$\xi^3_{yy} + \xi^3_{xx} = 0. \tag{2.97n}$$

From (2.97h) we get

$$\xi^3 = -y\xi^2_z + e^1(x), \tag{2.98}$$

where $e^1(x)$ is an arbitrary function. Equation (2.98) updates equations (2.97) as follows

$$y\xi^2_{zz} = 0, \tag{2.99a}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.99b}$$
$$e^1_x + \xi^1_z - y\xi^2_{xz} = 0, \tag{2.99c}$$
$$v\alpha K_1 + uK_2 + vK_4 + \alpha a + d - vK_1 - vK_3 - a_{zz} - a_{yy} - a_{xx} = 0, \tag{2.99d}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.99e}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.99f}$$
$$-e^1_{xx} + y\left(\xi^2_{zzz} + \xi^2_{xxz}\right) = 0, \tag{2.99g}$$

$$y\xi^2_{zz} = 0, \tag{2.99h}$$
$$\xi^1_y + \xi^2_x = 0, \tag{2.99i}$$
$$e^1_x + \xi^1_z - y\xi^2_{xz} = 0, \tag{2.99j}$$
$$vK_2 + u\alpha K_2 + d_{zz} + d_{yy} + d_{xx} - uK_2 - d = 0, \tag{2.99k}$$
$$\xi^1_{zz} + \xi^1_{yy} = 0, \tag{2.99l}$$
$$\xi^2_{zz} + \xi^2_{xx} = 0, \tag{2.99m}$$
$$-e^1_{xx} + y\left(\xi^2_{zzz} + \xi^2_{xxz}\right) = 0. \tag{2.99n}$$

Equations (2.99a,h) show that ξ^2 is linear in z, i.e

$$\xi^2 = e^2(x) + xe^3(x), \tag{2.100}$$

where e^2 and e^3 are arbitrary functions of x. Substituting (2.100) into (2.99b,i) and solving for ξ^1 yields

$$\xi^1 = y(-e^2_x - ze^3_x) + e^4(z), \tag{2.101}$$

where e^4 is an arbitrary function of z. Substitution of (2.100) and (2.101) into (2.99) leads to

$$y\left(e^2_{xx} + ze^3_{xx}\right) = 0, \tag{2.102a}$$
$$y\left(e^2_{xx} + ze^3_{xx}\right) = 0, \tag{2.102b}$$
$$e^1_x + e^4_z - 2ye^3_x = 0, \tag{2.102c}$$
$$\begin{aligned} vK_1 + vK_3 + \left(\frac{3vy}{2} + 2uy\alpha\right)e^2_{xx} + \frac{1}{2}yz(3v + 4u\alpha)e^3_{xx} + a_{zz} + a_{yy} + a_{xx} \\ -v\alpha K_1 - uK_2 - vK_4 - \alpha a - d = 0, \end{aligned} \tag{2.102d}$$
$$-e^4_{zz} + y\left(e^2_{xxx} + ze^3_{xxx}\right) = 0, \tag{2.102e}$$
$$e^2_{xx} + ze^3_{xx} = 0, \tag{2.102f}$$
$$e^1_{xx} - ye^3_{xx} = 0, \tag{2.102g}$$
$$y\left(e^2_{xx} + ze^3_{xx}\right) = 0, \tag{2.102h}$$
$$e^1_x + e^4_z - 2ye^3_x = 0, \tag{2.102i}$$
$$vK_2 + u\alpha K_2 + \frac{1}{2}vye^2_{xxxx} + \frac{1}{2}vyze^3_{xxxx} + d_{zz} + d_{yy} + d_{xx} - uK_2 - d = 0, \tag{2.102j}$$
$$-e^4_{zz} + 2y\left(e^2_{xxx} + ze^3_{xxx}\right) = 0, \tag{2.102k}$$
$$e^1_{xx} - 2ye^3_{xx} = 0. \tag{2.102l}$$

Separating (2.102l) by y shows that e^1 and e^3 are both linear in x, thus

$$e^1 = K_5 + xK_6, \tag{2.103a}$$
$$e^3 = K_7 + xK_8, \tag{2.103b}$$

for some arbitrary constants $K_5, \ldots, K_8$. Substituting (2.103a,b) into (2.103c,i) solves to

$$e^4 = z(2yK_8 - K_6) + K_9. \tag{2.104}$$

Substituting (2.103b) into (2.102a,b,f,h) implies that e^2 is linear in x, i.e.,

$$e^2 = K_9 + xK_{10}. \tag{2.105}$$

Substituting (2.103a,b), (2.104) and (2.105) into equations (2.101) we find that $K_2 = K_8 = 0$ and $K_4 = (1-\alpha)K_1 + K_3$. Making these substitutions satisfy all other equations and leave only

$$a_{xx} + a_{yy} + a_{zz} = \alpha a + d, \tag{2.106a}$$

$$d_{xx} + d_{yy} + d_{zz} = d, \tag{2.106b}$$

which is (2.91). Thus we have the solutions

$$\xi^1 = -zK_5 + K_7 - yK_{10}, \tag{2.107a}$$

$$\xi^2 = zK_6 + K_9 + xK_{10}, \tag{2.107b}$$

$$\xi^3 = K_4 + xK_5 - yK_6, \tag{2.107c}$$

$$\eta^1 = vK_1 + uK_3 + a(x,y,z), \tag{2.107d}$$

$$\eta^2 = v(K_1 - \alpha K_1 + K_3) + d(x,y,z). \tag{2.107e}$$

Therefore, the symmetries of the system (2.91) are

$$X_1 = \partial_x, \tag{2.108a}$$

$$X_2 = \partial_y, \tag{2.108b}$$

$$X_3 = \partial_z, \tag{2.108c}$$

$$X_4 = y\partial_x - x\partial_y, \tag{2.108d}$$

$$X_5 = z\partial_x - x\partial_z, \tag{2.108e}$$

$$X_6 = z\partial_y - y\partial_z, \tag{2.108f}$$

$$X_7 = u\partial_u + v\partial_v, \tag{2.108g}$$

$$X_8 = v\partial_u + (1-\alpha)v\partial_v, \tag{2.108h}$$

$$X_a = a(x,y,z)\partial_u, \tag{2.108i}$$

$$X_b = d(x,y,z)\partial_v, \tag{2.108j}$$

where $a(x,y,z)$ and $d(x,y,z)$ solve system (2.91) or (2.106).

System (1.6g) in expanded form reads

$$u_{xx} + u_{yy} + u_{zz} = -\alpha u - v, \qquad \alpha \geq 0, \tag{2.109a}$$

$$v_{xx} + v_{yy} + v_{zz} = -v. \tag{2.109b}$$

This is system 7, it has the same symmetry Lie algebra as system 6 which is (2.108).

2.5 System 8

The expanded form of system (1.6g) is

$$u_{xx} + u_{yy} + u_{zz} = \alpha u + \beta v, \tag{2.110a}$$

$$v_{xx} + v_{yy} + v_{zz} = -\beta u + \alpha v, \qquad \alpha^2 + \beta^2 = 1. \tag{2.110b}$$

Proceeding likewise, system (2.110) admits the following symmetries

$$X_1 = \partial_x, \tag{2.111a}$$

$$X_2 = \partial_y, \tag{2.111b}$$

$$X_3 = \partial_z, \tag{2.111c}$$

$$X_4 = y\partial_x - x\partial_y, \tag{2.111d}$$

$$X_5 = z\partial_x - x\partial_z, \tag{2.111e}$$

$$X_6 = z\partial_y - y\partial_z, \tag{2.111f}$$

$$X_7 = u\partial_u + v\partial_v, \tag{2.111g}$$

$$X_8 = v\partial_u - u\partial_v, \tag{2.111h}$$

$$X_a = a(x, y, z)\partial_u, \tag{2.111i}$$

$$X_d = d(x, y, z)\partial_v, \tag{2.111j}$$

where $a(x, y, z)$ and $d(x, y, z)$ are solution to system (2.110).

Chapter 3

Similarity reductions of Elliptic Systems

In this chapter we construct the optimal system of subalgebras and the invariant solutions for elliptic systems investigated in Chapter 2. The original system is first reduced by its symmetry into an invariant system with two independent variables. Thereafter, the optimal system of the reduced elliptic system equations is used to perform similarity reductions and invariant solutions.

The main motivation for calculating the symmetries of the DEs is to use them to discover the structure of the solution. A symmetry has the interesting property of mapping solutions to not the same one. When dealing with PDEs, finding the analytic solution is something very difficult or impossible. However, symmetries can be used to reduce a PDE to an ODE which is generally easier to solve. Invariant solutions of a given equation satisfy an equation with fewer variables. If the reduction leads to another PDE, then the symmetries of the reduced PDE are used to perform further reductions.

Definition 3.0.1 The commutator of X_i and X_j is $[X_i, X_j] = X_i(X_j) - X_j(X_i)$. The requirement under here is that given a Lie algebra M (i.e. the vector space M of operators), $[X_i, X_j] \in M$ for all $X_i, X_j \in M$. For any operators $X_i, X_j, X_k \in M$ and for constants c_1 and c_2, we have the following properties of the commutators

1. Bilinear: $[c_1 X_i + c_2 X_j, X_k] = c_1[X_i, X_k] + c_2[X_j, X_k]$ and $[X_i, c_1 X_j + c_2 X_k] = c_1[X_i, X_j] + c_2[X_i, X_k]$,
2. Skew-Symmetry: $[X_i, X_j] = -[X_j, X_i]$,
3. Jacobi identity: $[[X_i, X_j], X_k] + [[X_j, X_k], X_i] + [[X_k, X_i], X_j] = 0$.

Definition 3.0.2 The adjoint representation is given by the formula

$$Ad(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \frac{\epsilon^3}{3!}[X_i[X_i[X_i, X_j]]] + \cdots.$$

3.1 System 1 or (2.1)

a) Reduction by X_4 yields the characteristic equations

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{dz}{0} = \frac{du}{0} = \frac{dv}{0}, \tag{3.1}$$

which give the invariants

$$z_1 = x^2 + y^2,\ z_2 = z,\ u = w^1(z_1, z_2),\ v = w^2(z_1, z_2). \tag{3.2}$$

Solutions (3.2) are substituted into (2.1) yielding the system of PDEs

$$4z_1 w^1_{z_1 z_1} + 4w^1_{z_1} + w^1_{z_2 z_2} = w^1, \tag{3.3a}$$

$$4z_1 w^2_{z_1 z_1} + 4w^2_{z_1} + w^2_{z_2 z_2} = \alpha w^2. \tag{3.3b}$$

System (3.3) has to be reduced further to a system of ODEs, to do so we have to first find its symmetries which are

$$Y_1 = \partial_{z_2},\ Y_2 = w^1\partial_{w^1},\ Y_3 = w^2\partial_{w^2},\ Y_a = a(z_1, z_2)\partial_{w^1},\ Y_b = b(z_1, z_2)\partial_{w^2}. \tag{3.4}$$

The commutator table for Lie algebra (3.4) is shown in Table 3.1.

Table 3.1: Commutator table

↗	Y_1	Y_2	Y_3
Y_1	0	0	0
Y_2	0	0	0
Y_3	0	0	0

With the help of Table 3.1, the adjoint representation table for Lie algebra (3.4) is presented in Table 3.2.

Table 3.2: Adjoint representation table

↗	Y_1	Y_2	Y_3
Y_1	Y_1	Y_2	Y_3
Y_2	Y_1	Y_2	Y_3
Y_3	Y_1	Y_2	Y_3

Now we find the optimal system for the finite part of the symmetry Lie algebra (3.4), i.e., we consider symmetries Y_1, Y_2 and Y_3.

Let

$$Y = b_1 Y_1 + b_2 Y_2 + b_3 Y_3, \tag{3.5}$$

where b_i's are arbitrary constants which cannot be zero simultaneously. With the help of adjoint table, it can be seen that Y cannot be reduced by the adjoint action. Hence the optimal system for system (3.3) consists of linear combination of (3.5). We consider cases for $b_i = 0, 1$ and thus the outcome is

$$\{Y_1,\ Y_2,\ Y_3,\ Y_1 + Y_2,\ Y_1 + Y_3,\ Y_2 + Y_3,\ Y_1 + Y_2 + Y_3\}. \tag{3.6}$$

Only the subalgebras that leave system (3.3) invariant are considered. Throughout the calculations, K_i's are considered arbitrary constants.

i. Invariance under Y_1. The characteristic equations are

$$\frac{dz_1}{0} = \frac{dz_2}{1} = \frac{dw^1}{0} = \frac{dw^2}{0}, \tag{3.7}$$

which give the solutions

$$z^* = z_1,\ w^1 = F(z^*),\ w^2 = G(z^*). \tag{3.8}$$

Equations (3.8) are substituted into the system (3.3) to give a system of ODEs with variable coefficients

$$4z^* F''(z^*) + 4F'(z^*) = F(z^*), \tag{3.9a}$$

$$4z^* G''(z^*) + 4G'(z^*) = \alpha G(z^*). \tag{3.9b}$$

We determine the power series solution of system (3.9) about the origin. Firstly consider (3.9a), $z_0^* = 0$ is a regular singular point, so we assume a solution of the form

$$F(z^*) = \sum_{m=0}^{\infty} a_m {z^*}^{m+r}, a_0 \neq 0, \tag{3.10}$$

Substitution of equation (3.10) into (3.9a) yields

$$4\sum_{m=2}^{\infty}(m+r)(m+r-1)a_m {z^*}^{m+r-1} + 4\sum_{m=1}^{\infty}(m+r)a_m {z^*}^{m+r-1} - \sum_{m=0}^{\infty} a_m {z^*}^{m+r} = 0. \tag{3.11}$$

Solving (3.11) we obtain

$$F(z^*) = K_1 F_1(z^*) + K_2 F_2(z^*), \tag{3.12}$$

where

$$F_1(z^*) = a_0 \sum_{n=0}^{\infty} \frac{{z^*}^n}{4^n (n!)^2}, \tag{3.13a}$$

$$F_2(z^*) = F_1(z^*)\ln(z^*) + b_0 + b_1 z^* + b_2 {z^*}^2 + \cdots, \tag{3.13b}$$

and

$$b_1 = \frac{b_0}{4(1)^2} - \frac{8a_0}{4^3(1!)^2(1)}, \tag{3.14a}$$

$$b_2 = \frac{b_1}{4(2)^2} - \frac{8a_0}{4^4(2!)^2(2)}. \tag{3.14b}$$

Through back substitution, we find the invariant solution

$$\begin{aligned} u(x,y,z) =& K_1 \sum_{n=0}^{\infty} \frac{a_0(x^2+y^2)^n}{4^n(n!)^2} \\ &+ K_2\Big[\sum_{n=0}^{\infty} \frac{a_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2) + b_0 + b_1(x^2+y^2) + b_2(x^2+y^2)^2 + \cdots\Big]. \end{aligned} \tag{3.15}$$

Similarly, invariant solution with respect to (3.9b) is

$$\begin{aligned} v(x,y,z) =& K_3 \sum_{n=0}^{\infty} \frac{a_0\alpha^n(x^2+y^2)^n}{4^n(n!)^2} \\ &+ K_4\Big[\sum_{n=0}^{\infty} \frac{a_0\alpha^n(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2) + c_0 + c_1(x^2+y^2) + c_2(x^2+y^2)^2 + \cdots\Big]. \end{aligned} \tag{3.16}$$

where

$$c_1 = \frac{\alpha c_0}{4(1)^2} - \frac{8a_0}{4^3(1!)^2(1)}, \tag{3.17a}$$

$$c_2 = \frac{\alpha c_1}{4(2)^2} - \frac{8a_0}{4^4(2!)^2(2)}. \tag{3.17b}$$

ii. Invariance under $Y_1 + Y_2$. The characteristic equations are

$$\frac{dz_1}{0} = \frac{dz_2}{1} = \frac{dw^1}{w^1} = \frac{dw^2}{0}, \tag{3.18}$$

which have solutions

$$z^* = z_1,\ w^1 = e^{z_2}F(z^*),\ w^2 = G(z^*). \tag{3.19}$$

Invariants (3.19) reduce system (3.3) to a system of ODEs

$$z^*F''(z^*) + F'(z^*) = 0, \tag{3.20a}$$

$$4z^*G''(z^*) + 4G'(z^*) = \alpha G. \tag{3.20b}$$

Equation (3.20a) solves to

$$F(z^*) = K_5\ln(z^*) + K_6. \tag{3.21}$$

By back substitutions, (3.21) yields invariant solution

$$u(x,y,z) = e^z\Big(K_5\ln(x^2+y^2) + K_6\Big). \tag{3.22}$$

Equation (3.20b) solves to

$$\begin{aligned} v(x,y,z) =& K_7\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+y^2)^n}{4^n(n!)^2} \\ &+ K_8\Big[\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2) + c_0 + c_1(x^2+y^2) + c_2\big(x^2+y^2\big)^2 + \cdots\Big]. \end{aligned} \tag{3.23}$$

iii. Invariance under $Y_1 + Y_3$. The characteristic equations

$$\frac{dz_1}{0} = \frac{dz_2}{1} = \frac{dw^1}{0} = \frac{dw^2}{w^2}, \tag{3.24}$$

yield the invariants

$$z^* = z_1,\ w^1 = F(z^*),\ w^2 = e^{z_2}G(z^*). \tag{3.25}$$

Solutions (3.25) reduce (3.3) to a system of ODEs

$$4z^*F''(z^*) + 4F'(z^*) - F(z^*) = 0, \tag{3.26a}$$

$$4z^*G''(z^*) + 4G'(z^*) + (1-\alpha)G(z^*) = 0. \tag{3.26b}$$

The invariant solutions for system (3.26) are

$$\begin{aligned} u(x,y,z) =& K_7\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)}{4^n(n!)^2} \\ &+ K_8\Big[\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)}{4^n(n!)^2}\ln(x^2+y^2) + b_0 + b_1(x^2+y^2) + b_2\big(x^2+y^2\big)^2 + \cdots\Big], \end{aligned} \tag{3.27a}$$

$$v(x,y,z)=e^{z}\Big(K_9\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+y^2)}{4^n(n!)^2} + K_{10}\Big[\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+y^2)}{4^n(n!)^2}\ln(x^2+y^2)+c_0+c_1(x^2+y^2)+c_2\big(x^2+y^2\big)^2+\cdots\Big]\Big). \tag{3.27b}$$

iv. Invariance under $Y_1+Y_2+Y_3$. The characteristic equations

$$\frac{dz_1}{0}=\frac{dz_2}{1}=\frac{dw^1}{w^1}=\frac{dw^2}{w^2}, \tag{3.28}$$

solve to the following invariants

$$z^*=z_1,\ w^1=e^{z_2}F(z^*),\ w^2=e^{z_2}G(z^*). \tag{3.29}$$

Invariants (3.29) reduce system (3.3) to a system of ODEs

$$z^*F''(z^*)+F'(z^*)=0, \tag{3.30a}$$

$$4z^*G''(z^*)+4G'(z^*)+(1-\alpha)G=0. \tag{3.30b}$$

From system (3.30), we get the invariant solutions

$$u(x,y,z)=e^{z}\Big(K_{11}\ln(x^2+y^2)+K_{12}\Big), \tag{3.31a}$$

$$v(x,y,z)=e^{z}\Big(K_{13}\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+y^2)}{4^n(n!)^2} + K_{14}\Big[\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+y^2)}{4^n(n!)^2}\ln(x^2+y^2)+c_0+c_1(x^2+y^2)+c_2\big(x^2+y^2\big)^2+\cdots\Big]\Big). \tag{3.31b}$$

Using X_5 and X_6, we can reduce system (2.1) in the similar way we did using X_4. These symmetries reduce system (2.1) to a system (3.3) and hence leads to similar symmetries Y_1, Y_2, Y_3. Tables 3.3 and 3.4 summarize invariant solutions due to linear combinations of Y_1, Y_2, Y_3 under X_5 and X_6 respectively.

Table 3.3: Invariant solutions using X_5.

Subalgebra	Reduced System	Invariant Solution
Y_1	$4z^*F''(z^*)+4F'(z^*)=F(z^*)$,	$u(x,y,z)=K_1\sum_{n=0}^{\infty}\frac{a_0(x^2+z^2)^n}{4^n(n!)^2}$ $+K_2\Big[\sum_{n=0}^{\infty}\frac{a_0(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+b_0+b_1(x^2+z^2)+b_2\big(x^2+z^2\big)^2+\cdots\Big]$,
	$4z^*G''(z^*)+4G'(z^*)=\alpha G(z^*)$.	$v(x,y,z)=K_3\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+z^2)^n}{4^n(n!)^2}$ $+K_4\Big[\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+c_0+c_1(x^2+z^2)+c_2\big(x^2+z^2\big)^2+\cdots\Big]$.
Y_1+Y_2	$z^*F''(z^*)+F'(z^*)=0$,	$u(x,y,z)=e^y\big(K_5\ln(x^2+z^2)+K_6\big)$,
	$4z^*G''(z^*)+4G'(z^*)=\alpha G(z^*)$.	$v(x,y,z)=K_7\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+z^2)^n}{4^n(n!)^2}$ $+K_8\Big[\sum_{n=0}^{\infty}\frac{a_0\alpha^n(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+c_0+c_1(x^2+z^2)+c_2\big(x^2+z^2\big)^2+\cdots\Big]$.
Y_1+Y_3	$4z^*F''(z^*)+4F'(z^*)=F(z^*)$,	$u(x,y,z)=K_9\sum_{n=0}^{\infty}\frac{a_0(x^2+z^2)^n}{4^n(n!)^2}$ $+K_{10}\Big[\sum_{n=0}^{\infty}\frac{a_0(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+b_0+b_1(x^2+z^2)+b_2\big(x^2+z^2\big)^2+\cdots\Big]$,
	$4z^*G''(z^*)+4G'(z^*)+(1-\alpha)G(z^*)=0$.	$v(x,y,z)=K_{11}\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+z^2)^n}{4^n(n!)^2}$ $+K_{12}\Big[\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+c_0+c_1(x^2+z^2)+c_2\big(x^2+z^2\big)^2+\cdots\Big]$.
$Y_1+Y_2+Y_3$	$z^*F''(z^*)+F'(z^*)=0$,	$u(x,y,z)=e^y\big(K_{13}\ln(x^2+z^2)+K_{14}\big)$,
	$4z^*G''(z^*)+4G'(z^*)+(1-\alpha)G(z^*)=0$.	$v(x,y,z)=K_{15}\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+z^2)^n}{4^n(n!)^2}$ $+K_{16}\Big[\sum_{n=0}^{\infty}\frac{a_0(1-\alpha)^n(x^2+z^2)^n}{4^n(n!)^2}\ln(x^2+z^2)+c_0+c_1(x^2+z^2)+c_2\big(x^2+z^2\big)^2+\cdots\Big]$.

Table 3.4: Invariant solutions using X_6.

Subalgebra	Reduced System	Invariant Solution
Y_1	$4z^* F''(z^*) + 4F'(z^*) = F(z^*),$	$u(x,y,z) = K_1 \sum_{n=0}^{\infty} \frac{a_0 (y^2+z^2)^n}{4^n (n!)^2} + K_2 \Big[\sum_{n=0}^{\infty} \frac{a_0 (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + b_0 + b_1(y^2+z^2) + b_2\big(y^2+z^2\big)^2 + \cdots \Big],$
	$4z^* G''(z^*) + 4G'(z^*) = \alpha G(z^*).$	$v(x,y,z) = K_3 \sum_{n=0}^{\infty} \frac{a_0 \alpha^n (y^2+z^2)^n}{4^n (n!)^2} + K_4 \Big[\sum_{n=0}^{\infty} \frac{a_0 \alpha^n (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + c_0 + c_1(y^2+z^2) + c_2\big(y^2+z^2\big)^2 + \cdots \Big].$
$Y_1 + Y_2$	$z^* F''(z^*) + F'(z^*) = 0,$	$u(x,y,z) = e^x\big(K_5 \ln(y^2+z^2) + K_6\big),$
	$4z^* G''(z^*) + 4G'(z^*) = \alpha G(z^*).$	$v(x,y,z) = K_7 \sum_{n=0}^{\infty} \frac{a_0 \alpha^n (y^2+z^2)^n}{4^n (n!)^2} + K_8 \Big[\sum_{n=0}^{\infty} \frac{a_0 \alpha^n (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + c_0 + c_1(y^2+z^2) + c_2\big(y^2+z^2\big)^2 + \cdots \Big].$
$Y_1 + Y_3$	$4z^* F''(z^*) + 4F'(z^*) = F(z^*),$	$u(x,y,z) = K_9 \sum_{n=0}^{\infty} \frac{a_0 (y^2+z^2)^n}{4^n (n!)^2} + K_{10} \Big[\sum_{n=0}^{\infty} \frac{a_0 (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + b_0 + b_1(y^2+z^2) + b_2\big(y^2+z^2\big)^2 + \cdots \Big],$
	$4z^* G''(z^*) + 4G'(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = K_{11} \sum_{n=0}^{\infty} \frac{a_0 (1-\alpha)^n (y^2+z^2)^n}{4^n (n!)^2} + K_{12} \Big[\sum_{n=0}^{\infty} \frac{a_0 (1-\alpha)^n (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + c_0 + c_1(y^2+z^2) + c_2\big(y^2+z^2\big)^2 + \cdots \Big].$
$Y_1 + Y_2 + Y_3$	$z^* F''(z^*) + F'(z^*) = 0,$	$u(x,y,z) = e^x\big(K_{13} \ln(y^2+z^2) + K_{14}\big),$
	$4z^* G''(z^*) + 4G'(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = K_{15} \sum_{n=0}^{\infty} \frac{a_0 (1-\alpha)^n (y^2+z^2)^n}{4^n (n!)^2} + K_{16} \Big[\sum_{n=0}^{\infty} \frac{a_0 (1-\alpha)^n (y^2+z^2)^n}{4^n (n!)^2} \ln(y^2+z^2) + c_0 + c_1(y^2+z^2) + c_2\big(y^2+z^2\big)^2 + \cdots \Big].$

b) Reduction by $X_1+X_7+X_8$ yields characteristic equations

$$\frac{dx}{1}=\frac{dy}{0}=\frac{dz}{0}=\frac{du}{u}=\frac{dv}{v}, \tag{3.32}$$

and hence the invariants

$$z_1=y,\, z_2=z,\, u=e^x w^1(z_1,z_2),\;\, v=e^x w^2(z_1,z_2). \tag{3.33}$$

Solutions (3.33) are substituted into system (2.1) yielding the system of PDEs

$$w^1_{z_1z_1}+w^1_{z_2z_2}=0, \tag{3.34a}$$

$$w^2_{z_1z_1}+w^2_{z_2z_2}+(1-\alpha)w^2=0, \tag{3.34b}$$

which has symmetries

$$\begin{gathered} Y_1=\partial_{z_1},\; Y_2=\partial_{z_2},\; Y_3=z_2\partial_{z_1}-z_1\partial_{z_2},\; Y_4=w^1\partial_{w^1},\\ Y_5=w^2\partial_{w^2},\; Y_a=a(z_1,z_2)\partial_{w^1},\; Y_b=b(z_1,z_2)\partial_{w^2}. \end{gathered} \tag{3.35}$$

Commutator and adjoint representation tables for Lie algebra (4.20) look thus

Table 3.5: Commutator table

$\nearrow$	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	0	0	$-Y_2$	0	0
Y_2	0	0	Y_1	0	0
Y_3	Y_2	$-Y_1$	0	0	0
Y_4	0	0	0	0	0
Y_5	0	0	0	0	0

Table 3.6: Adjoint representation table

Ad	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	Y_1	Y_2	$Y_3+\epsilon Y_2$	Y_4	Y_5
Y_2	Y_1	Y_2	$Y_3-\epsilon Y_1$	Y_4	Y_5
Y_3	$Y_1\cos\epsilon-Y_2\sin\epsilon$	$Y_2\cos\epsilon+Y_1\sin\epsilon$	Y_3	Y_4	Y_5
Y_4	Y_1	Y_2	Y_3	Y_4	Y_5
Y_5	Y_1	Y_2	Y_3	Y_4	Y_5

In constructing the optimal system of the finite symmetry group $\{Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5\}$, we consider the general operator

$$Y=a_1Y_1+a_2Y_2+a_3Y_3+a_4Y_4+a_5Y_5 \tag{3.36}$$

where $a_1,\cdots,a_5$ are arbitrary constants and try to reduce (3.36) by the adjoint action to a new operator having a simpler form. It should first be noticed that Y_4 and Y_5 are central elements so they can be disregarded preliminarily and added later to each of the found subalgebra as a direct sum with arbitrary constants. The operator (3.36) reduces to

$$Y=a_1Y_1+a_2Y_2+a_3Y_3. \tag{3.37}$$

First we let $a_3 \neq 0$, (take $a_3 = 1$) such that (3.37) becomes

$$Y = a_1 Y_1 + a_2 Y_2 + Y_3. \tag{3.38}$$

With the help of Table 3.6, acting upon (3.38) by $Ad(e^{-a_2 Y_1})$ eliminates $a_2 Y_2$ and hence (3.38) reduces to

$$Y' = a_1' Y_1 + Y_3. \tag{3.39}$$

From (3.39), $a_1' Y_1$ gets eliminated by the action of $Ad(e^{a_1' Y_2})$ on (3.39) and leaves a simplified subalgebra

$$Y'' = Y_3. \tag{3.40}$$

Next from (3.37) we let $a_3 = 0$ and $a_1 = a_2 = 1$ such that

$$Y = Y_1 + Y_2. \tag{3.41}$$

Upon acting on (3.41) by $Ad(e^{\frac{\pi}{4} Y_3})$ eliminates $a_2 Y_2$ and thus we have

$$Y' = \sqrt{2} Y_1, \tag{3.42}$$

which cannot simplify further. Finally letting $a_1 = a_3 = 0$ from (3.37) leaves

$$Y = a_2 Y_2. \tag{3.43}$$

From (3.40),(3.42) and (3.43) and also considering the central elements, we find the following optimal system

$$\left\{Y_1 + \lambda Y_4 + \mu Y_5,\ Y_2 + \lambda Y_4 + \mu Y_5,\ Y_3 + \lambda Y_4 + \mu Y_5\right\}, \tag{3.44}$$

for some arbitrary constants λ, μ.

As an example, invariance under Y_1 is worked out fully, the rest of the combinations are shown in Table 3.7. K_i's represent arbitrary constants.

Invariance under Y_1: The characteristic equations

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dw^1}{0} = \frac{dw^2}{0}, \tag{3.45}$$

give the invariants

$$z^* = z_2,\ w^1 = F(z^*),\ w^2 = G(z^*). \tag{3.46}$$

Solutions (3.46) substituted into system (3.34) yield a system of ODEs

$$F''(z^*) = 0, \tag{3.47a}$$

$$G''(z^*) + (1-\alpha)G(z^*) = 0, \tag{3.47b}$$

which solves to

$$F(z^*) = K_1 z^* + K_2, \tag{3.48a}$$

$$G(z^*) = K_3 \cos\left(\sqrt{1-\alpha} z^*\right) + K_4 \sin\left(\sqrt{1-\alpha} z^*\right). \tag{3.48b}$$

Through back substitutions we get invariant solutions

$$u(x,y,z) = e^x\left(K_1 z + K_2\right), \tag{3.49a}$$

$$v(x,y,z) = e^x\left(K_3 \cos\left(z\sqrt{1-\alpha}\right) + K_4 \sin\left(z\sqrt{1-\alpha}\right)\right). \tag{3.49b}$$

Table 3.7: Invariant solutions of Lie algebra (4.20)

Subalgebra	Reduced System	Invariant Solution
Y_2	$F''(z^*) = 0,$	$u(x,y,z) = e^x\big(K_3 z + K_4\big),$
	$G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^x\big(K_5 e^{z\sqrt{1-\alpha}} + K_6 e^{-z\sqrt{1-\alpha}}\big).$
Y_3	$z^* F''(z^*) + F'(z^*) = 0,$	$u(x,y,z) = e^x\big(K_7 \ln(y^2+z^2) + K_8\big),$
	$4z^* G''(z^*) + 4G'(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^x\Big(K_9 \sum_{n=0}^{\infty} \frac{a_0(1-\alpha)^n(y^2+z^2)^n}{4^n(n!)^2}$ $+K_{10}\Big[\sum_{n=0}^{\infty} \frac{a_0(1-\alpha)^n(y^2+z^2)^n}{4^n(n!)^2} \ln(y^2+z^2) + c_0 + c_1(y^2+z^2) + c_2\big(y^2+z^2\big)^2 + \cdots\Big]\Big).$
$Y_1 + Y_2$	$F''(z^*) = 0,$	$u(x,y,z) = e^x\Big(K_{11}(y-z) + K_{12}\Big),$
	$2G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^x\Big(K_{13} \exp\Big\{\sqrt{\frac{1-\alpha}{2}}(y-z)\Big\} K_{14} \exp\Big\{-\sqrt{\frac{1-\alpha}{2}}(y-z)\Big\}\Big).$
$Y_1 + Y_2 + Y_4$	$2F''(z^*) + 2F'(z^*) + F(z^*) = 0,$	$u(x,y,z) = e^{-\frac{xy}{2}(y-z)}\Big(K_{15}\cos\big(\frac{y-z}{2}\big) + K_{16}\sin\big(\frac{y-z}{2}\big)\Big),$
	$2G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^x\Big(K_{17}\cos\Big(\sqrt{\frac{1-\alpha}{2}}(y-z)\Big) + K_{18}\sin\Big(\sqrt{\frac{1-\alpha}{2}}(y-z)\Big)\Big).$
$Y_1 + Y_2 + Y_5$	$F''(z^*) = 0,$	$u(x,y,z) = e^x\Big(K_{19}(y-z) + K_{20}\Big),$
	$2G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{-\frac{xy}{2}(y-z)}\Big(K_{21}\cos\sqrt{\frac{1-2\alpha}{2}}(y-z) + K_{22}\sin\sqrt{\frac{1-2\alpha}{2}}(y-z)\Big).$

Subalgebra	Reduced System	Invariant Solution
$Y_1 + Y_4$	$F''(z^*) + F(z^*) = 0,$	$u(x,y,z) = e^{xy}\Big(K_{23}\cos(z) + K_{24}\sin(z)\Big),$
	$G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{x}\Big(K_{25}e^{z\sqrt{1-\alpha}} + K_{26}e^{-z\sqrt{1-\alpha}}\Big).$
$Y_1 + Y_5$	$F''(z^*) = 0,$	$u(x,y,z) = e^{x}\big(K_{27}\ln(y^2+z^2) + K_{28}\big),$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{xy}\Big(K_{29}e^{\sqrt{2-\alpha}z} + K_{30}e^{-\sqrt{2-\alpha}z}\Big).$
$Y_1 + Y_4 + Y_5$	$F''(z^*) + F(z^*) = 0,$	$u(x,y,z) = e^{xy}\Big(K_{31}\cos(z) + K_{32}\sin(z)\Big),$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{xy}\Big(K_{33}e^{z\sqrt{2-\alpha}} + K_{34}e^{z\sqrt{2-\alpha}}\Big).$
$Y_2 + Y_4 + Y_5$	$F''(z^*) + F(z^*) = 0,$	$u(x,y,z) = e^{xz}\Big(K_{35}\cos(y) + K_{36}\sin(y)\Big).$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{xz}\Big(K_{37}e^{y\sqrt{2-\alpha}} + K_{38}e^{y\sqrt{2-\alpha}}\Big).$
$Y_2 + Y_4$	$F''(z^*) + F(z^*) = 0,$	$u(x,y,z) = e^{xz}\Big(K_{39}\cos(y) + K_{40}\sin(y)\Big),$
	$G''(z^*) + (1-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{x}\Big(K_{41}e^{y\sqrt{1-\alpha}} + K_{42}e^{-y\sqrt{1-\alpha}}\Big).$

Subalgebra	Reduced System	Invariant Solution
$Y_1+Y_2+Y_4+Y_5$	$2F''(z^*)+2F'(z^*)+F(z^*)=0,$	$u(x,y,z)=e^{-\frac{1}{2}xy(y-z)}\Big(K_{47}\cos\big(\frac{1}{2}(y-z)\big)+K_{48}\sin\big(\frac{1}{2}(y-z)\big)\Big),$
	$2G''(z^*)+2G'(z^*)+(2-\alpha)G(z^*)=0.$	$u(x,y,z)=e^{-\frac{1}{2}xy(y-z)}\Big(K_{49}\cos\big(\frac{2-\alpha}{2}(y-z)\big)+K_{50}\sin\big(\frac{2-\alpha}{2}(y-z)\big)\Big).$
Y_2+Y_5	$F''(z^*)=0,$	$u(x,y,z)=e^{x}\big(K_{51}\ln(y^2+z^2)+K_{52}\big),$
	$G''(z^*)+(2-\alpha)G(z^*)=0.$	$v(x,y,z)=e^{xz}\Big(K_{53}e^{y\sqrt{2-\alpha}}+K_{54}e^{-y\sqrt{2-\alpha}}\Big).$
$Y_1+Y_2+Y_4+Y_5$	$2F''(z^*)+2F'(z^*)+F(z^*)=0,$	$u(x,y,z)=e^{-\frac{1}{2}xy}\Big(K_{55}\cos\big(\frac{1}{2}(y-z)\big)+K_{56}\sin\big(\frac{1}{2}(y-z)\big)\Big),$
	$2G''(z^*)+2G'(z^*)+(2-\alpha)G(z^*)=0.$	$u(x,y,z)=e^{-\frac{1}{2}xy}\Big(K_{57}\cos\big(\frac{2-\alpha}{2}(y-z)\big)+K_{58}\sin\big(\frac{2-\alpha}{2}(y-z)\big)\Big).$

To find invariant solutions for the other systems, we follow the similar steps.

3.2 System 2 or (2.33)

It was stated in Chapter 2 that symmetry group for this system is similar to that of system (2.1). To construct the invariant solutions under here we proceed as in the previous case. We will derive invariant solutions by first reducing system (2.33) using reduction by the linear combination $X_3 + X_7 + X_8$, the characteristic equations are

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dz}{1} = \frac{du}{1} = \frac{dv}{1}. \tag{3.50}$$

The corresponding invariants are

$$z_1 = x,\ z_2 = y,\ u = e^z w^1(z_1, z_2),\ v = e^z w^2(z_1, z_2). \tag{3.51}$$

Substituting (3.51) into (2.33) yields the system

$$w^1_{z_1 z_1} + w^1_{z_2 z_2} + 2w^1 = 0, \tag{3.52a}$$

$$w^2_{z_1 z_1} + w^2_{z_2 z_2} + (1-\alpha) w^2 = 0. \tag{3.52b}$$

System (3.52) has symmetry Lie algebra

$$\begin{aligned} &Y_1 = \partial_{z_1},\ Y_2 = \partial_{z_2}, Y_3 = z_2\partial_{z_1} - z_1\partial_{z_2},\ Y_4 = w^1\partial_{w^1}, \\ &Y_5 = w^2\partial_{w^2},\ Y_a = a(z_1, z_2)\partial_{w^1},\ Y_b = b(z_1, z_2)\partial_{w^2}, \end{aligned} \tag{3.53}$$

which coincide with Lie algebra (4.20), hence the commutator table and the adjoint representation table for system (3.52) are presented in Tables 3.5 and 3.6 respectively. Also, the optimal system for system (3.52) is given by (3.44). Invariant solutions are calculated using the linear combination of Lie algebra (3.53). Table 3.8 shows the subalgebra, corresponding reduced system and invariant solution for system (2.33).

Table 3.8: Invariant solutions of Lie algebra (3.53)

Subalgebra	Reduced System	Invariant Solution
Y_1	$F''(z^*)+2F(z^*)=0,$ $G''(z^*)+(1-\alpha)G(z^*)=0.$	$u(x,y,z)=e^z\big(K_1\cos(\sqrt{2}y)+K_2\sin(\sqrt{2}y)\big),$ $v(x,y,z)=e^z\big(K_3\cos(\sqrt{1-\alpha}\,y)K_4\sin(\sqrt{1-\alpha}\,y)\big).$
Y_2	$F''(z^*)+2F(z^*)=0,$ $G''(z^*)+(1-\alpha)G(z^*)=0.$	$u(x,y,z)=e^z\big(K_5\cos(\sqrt{2}x)+K_6\sin(\sqrt{2}x)\big),$ $v(x,y,z)=e^z\Big[K_7\cos(\sqrt{1-\alpha}\,x)+K_8\sin(\sqrt{1-\alpha}\,x)\Big]$
Y_3	$4z^*F''(z^*)+4F'(z^*)+2F(z^*)=0,$ $4z^*G''(z^*)+4G'(z^*)+(1-\alpha)G(z^*)=0.$	$u(x,y,z)=e^z\Big(K_9\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}$ $+K_{10}\Big[\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2)+b_0+b_1(x^2+y^2)+b_2\big(x^2+y^2\big)^2+\cdots\Big]\Big),$ $v(x,y,z)=e^z\Big(K_{11}\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}$ $+K_{12}\Big[\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2)+c_0+c_1(x^2+y^2)+c_2\big(x^2+y^2\big)^2+\cdots\Big]\Big).$
Y_1+Y_2	$F''(z^*)+F(z^*)=0,$ $2G''(z^*)+(1-\alpha)G(z^*)=0.$	$u(x,y,z)=e^z\Big(K_{13}\cos\big(x-y\big)+K_{14}\sin\big(x-y\big)\Big),$ $v(x,y,z)=e^z\Big(K_{15}\cos\Big(\sqrt{\tfrac{1-\alpha}{2}}(x-y)\Big)+K_{16}\sin\Big(\sqrt{\tfrac{1-\alpha}{2}}(x-y)\Big)\Big).$
Y_1+Y_4	$F''(z^*)+3F(z^*)=0,$ $2G''(z^*)+(1-\alpha)G(z^*)=0.$	$u(x,y,z)=e^{xz}\Big(K_{17}\cos\big(\sqrt{3}y\big)+K_{18}\sin\big(\sqrt{3}y\big)\Big),$ $v(x,y,z)=e^z\Big(K_{17}\cos\big(\sqrt{1-\alpha}\,y\big)+K_{18}\sin\big(\sqrt{1-\alpha}\,y\big)\Big)$

Subalgebra	Reduced System	Invariant Solution
$Y_1 + Y_5$	$F''(z^*) + 2F(z^*) = 0,$	$u(x,y,z) = e^{z}\Big(K_{19}\cos(\sqrt{2}y) + K_{20}\sin(\sqrt{2}y)\Big),$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{xz}\Big(K_{21}\cos(\sqrt{2-\alpha}\,y) + K_{22}\sin(\sqrt{2-\alpha}\,y)\Big).$
$Y_2 + Y_4$	$F''(z^*) + 3F(z^*) = 0,$	$u(x,y,z) = e^{yz}\big(K_{23}\cos(\sqrt{3}x) + K_{24}\sin(\sqrt{3}x)\big),$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{z}\big(K_{25}\cos(\sqrt{2-\alpha}\,x)K_{26}\sin(\sqrt{2-\alpha}\,x)\big).$
$Y_2 + Y_5$	$F''(z^*) + 3F(z^*) = 0,$	$u(x,y,z) = e^{z}\big(K_{27}\cos(\sqrt{3}x) + K_{6}\sin(\sqrt{3}x)\big),$
	$G''(z^*) + (2-\alpha)G(z^*) = 0.$	$v(x,y,z) = e^{yz}\Big(K_{29}\cos(\sqrt{2-\alpha}\,x) + K_{30}\sin(\sqrt{2-\alpha}\,x)\Big)$

3.3 System 3 or (2.34)

Reduction by $X_2 + X_{11}$ yields reduced system

$$w^1_{z_1 z_1} + w^1_{z_2 z_2} + w^1 = 0, \tag{3.54a}$$
$$w^2_{z_1 z_1} + w^2_{z_2 z_2} + w^2 = 0, \tag{3.54b}$$

which has symmetries

$$\begin{aligned} Y_1 = \partial_{z_1},\ Y_2 = \partial_{z_2},\ Y_3 = z_2\partial_{z_1} - z_1\partial_2,\ Y_4 = w^1\partial_{w^1} + w^2\partial_{w^2}, \\ Y_5 = w^2\partial_{w^1} + w^2\partial_{w^2}, Y_a = a(z_1, z_2)\partial_{w^1}, Y_b = b(z_1, z_2)\partial_{w^2}. \end{aligned} \tag{3.55}$$

Table 3.9 gives a full list of possible invariant solutions.

Table 3.9: Invariant solutions of Lie algebra (3.55)

Subalgebra	Reduced System	Invariant Solution
Y_1	$F''(z^*)+F(z^*)=0,$ $G''(z^*)+G(z^*)=0.$	$u(x,y,z)=e^{z}\Big(K_1\cos(y)+K_2\sin(y)\Big),$ $v(x,y,z)=e^{z}\Big(K_3\cos(y)+K_4\sin(y)\Big).$
Y_2	$F''(z^*)+F(z^*)=0,$ $G''(z^*)+G(z^*)=0.$	$u(x,y,z)=e^{z}\Big(K_5\cos(x)+K_6\sin(x)\Big),$ $v(x,y,z)=e^{z}\Big(K_7\cos(x)+K_8\sin(x)\Big).$
Y_3	$4z^*F''(z^*)+4F'(z^*)+F(z^*)=0,$ $4z^*G''(z^*)+4G'(z^*)+G(z^*)=0.$	$u(x,y,z)=K_9\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}$ $+K_{10}\Big(\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2)+b_0+b_1(x^2+y^2)+b_2\big(x^2+y^2\big)^2+\cdots\Big),$ $v(x,y,z)=K_{11}\sum_{n=0}^{\infty}\frac{c_0(x^2+y^2)^n}{4^n(n!)^2}$ $+K_{12}\Big(\sum_{n=0}^{\infty}\frac{c_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2)+d_0+d_1(x^2+y^2)+d_2\big(x^2+y^2\big)^2+\cdots\Big).$
Y_1+Y_4	$F''(z^*)+2F(z^*)=0,$ $G''(z^*)+2G(z^*)=0.$	$u(x,y,z)=e^{xz}\Big(K_{13}\cos(\sqrt{2}y)+K_{14}\sin(\sqrt{2}y)\Big),$ $v(x,y,z)=e^{xz}\Big(K_{15}\cos(\sqrt{2}y)+K_{16}\sin(\sqrt{2}y)\Big).$
Y_2+Y_4	$F''(z^*)+2F(z^*)=0,$ $G''(z^*)+2G(z^*)=0.$	$u(x,y,z)=e^{yz}\Big(K_{17}\cos(\sqrt{2}x)+K_{18}\sin(\sqrt{2}x)\Big),$ $v(x,y,z)=e^{yz}\Big(K_{19}\cos(\sqrt{2}x)+K_{20}\sin(\sqrt{2}x)\Big).$

3.4 System 4 or (2.67)

Reduction by X_6 leads to a system of PDEs

$$w^1_{z_1 z_1} + 4z_2 w^1_{z_2 z_2} + 4w^1_{z_2} = w^1, \tag{3.56a}$$
$$w^2_{z_1 z_1} + 4z_2 w^2_{z_2 z_2} + 4w^2_{z_2} = w^2. \tag{3.56b}$$

The symmetries of (3.56) are

$$Y_1 = \partial_{z_1}, Y_2 = \partial_{w^1}, Y_3 = \partial_{w^2}, Y_a = a(z_1, z_2)\partial_{w^1}, Y_b = b(z_1, z_2)\partial_{w^2}. \tag{3.57}$$

Invariant solutions corresponding to the Lie algebra (3.57) are presented in Table 3.10.

Table 3.10: Invariant solutions of Lie algebra (3.57)

Subalgebra	Reduced System	Invariant Solution
Y_1	$4z^*F''(z^*)+4F'(z^*)-F(z^*)=0,$	$u(x,y,z)=K_1\sum_{n=0}^{\infty}\frac{a_0(y^2+z^2)^n}{4^n(n!)^2} + K_2\Big(\sum_{n=0}^{\infty}\frac{a_0(y^2+z^2)^n}{4^n(n!)^2}\ln(y^2+z^2)+b_0+b_1(y^2+z^2)+b_2\big(y^2+z^2\big)^2+\cdots\Big),$
	$4z^*G''(z^*)+4G'(z^*)-G(z^*)=0.$	$v(x,y,z)=K_3\sum_{n=0}^{\infty}\frac{c_0(y^2+z^2)^n}{4^n(n!)^2} + K_4\Big(\sum_{n=0}^{\infty}\frac{c_0(y^2+z^2)^n}{4^n(n!)^2}\ln(y^2+z^2)+d_0+d_1(y^2+z^2)+d_2\big(y^2+z^2\big)^2+\cdots\Big).$
Y_1+Y_2	$z^*F''(z^*)+F'(z^*)=0,$	$u(x,y,z)=K_5+K_6\ln(y^2+z^2),$
	$4z^*G''(z^*)+4G'(z^*)-G(z^*)=0.$	$v(x,y,z)=K_7\sum_{n=0}^{\infty}\frac{c_0(y^2+z^2)^n}{4^n(n!)^2} + K_8\Big(\sum_{n=0}^{\infty}\frac{c_0(y^2+z^2)^n}{4^n(n!)^2}\ln(y^2+z^2)+d_0+d_1(y^2+z^2)+d_2\big(y^2+z^2\big)^2+\cdots\Big).$
Y_1+Y_3	$4z^*F''(z^*)+4F'(z^*)-F(z^*)=0,$	$u(x,y,z)=K_9\sum_{n=0}^{\infty}\frac{a_0(y^2+z^2)^n}{4^n(n!)^2} + K_{10}\Big(\sum_{n=0}^{\infty}\frac{a_0(y^2+z^2)^n}{4^n(n!)^2}\ln(y^2+z^2)+b_0+b_1(y^2+z^2)+b_2\big(y^2+z^2\big)^2+\cdots\Big),$
	$z^*G''(z^*)+G'(z^*)=0.$	$v(x,y,z)=K_{11}+K_{12}\ln(y^2+z^2).$

3.5 System 5 or (2.90)

Reducing system (2.90) by $X_3 + X_{10}$ yields the system

$$w^1_{z_1 z_1} + w^1_{z_2 z_2} + w^1 = 0, \quad (3.58a)$$

$$w^2_{z_1 z_1} + w^2_{z_2 z_2} + 2w^2 = 0, \quad (3.58b)$$

whose finite symmetries are

$$\begin{aligned} &Y_1 = \partial_{z_1},\ Y_2 = \partial_{z_2},\ Y_3 = z_2\partial_{z_1} - z_1\partial_{z_2},\ Y_4 = w^1\partial_{w^1} + w^2\partial_{w^2},\\ &Y_5 = w^2\partial_{w^1} + w^1\partial_{w^2},\ Y_a = a(z_1, z_2)\partial_{w^1},\ Y_b = b(z_1, z_2)\partial_{w^2}. \end{aligned} \quad (3.59)$$

Linear combinations of Lie algebra (3.59) lead to invariant solutions shown in Table 3.11.

Table 3.11: Invariant solutions of Lie algebra (3.59)

Subalgebra	Reduced System	Invariant Solution
Y_1	$F''(z^*)+F(z^*)=0,$	$u(x,y,z)=K_1\cos(y)+K_2\sin(y),$
	$G''(z^*)+2G(z^*)=0.$	$v(x,y,z)=e^z\big(K_3\cos(\sqrt{2}y)+K_4\sin(\sqrt{2}y)\big).$
Y_2	$F''(z^*)+F(z^*)=0,$	$u(x,y,z)=K_5\cos(x)+K_6\sin(x),$
	$G''(z^*)+2G(z^*)=0.$	$v(x,y,z)=e^z\big(K_7\cos(\sqrt{2}x)+K_8\sin(\sqrt{2}x)\big).$
Y_3	$4z^*F''(z^*)+4F'(z^*)+F(z^*)=0,$	$u(x,y,z)=K_9\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2} + K_{10}\Big(\sum_{n=0}^{\infty}\frac{a_0(x^2+y^2)^n}{4^n(n!)^2}\ln(x^2+y^2)+b_0+b_1(x^2+y^2)+b_2\big(x^2+y^2\big)^2+\cdots\Big),$
	$2z^*G''(z^*)+2G'(z^*)+G(z^*)=0.$	$v(x,y,z)=K_{11}\sum_{n=0}^{\infty}\frac{c_0(x^2+y^2)^n}{2^n(n!)^2} + K_{12}\Big(\sum_{n=0}^{\infty}\frac{c_0(x^2+y^2)^n}{2^n(n!)^2}\ln(x^2+y^2)+d_0+d_1(x^2+y^2)+d_2\big(x^2+y^2\big)^2+\cdots\Big).$
Y_1+Y_4	$F''(z^*)+2F(z^*)=0,$	$u(x,y,z)=e^x\big(K_{13}\cos(\sqrt{2}y)+K_{14}\sin(\sqrt{2}y)\big),$
	$G''(z^*)+3G(z^*)=0.$	$v(x,y,z)=e^{xz}\big(K_{15}\cos(\sqrt{3}y)+K_{16}\sin(\sqrt{3}y)\big).$
Y_2+Y_4	$F''(z^*)+2F(z^*)=0,$	$u(x,y,z)=e^y\big(K_{17}\cos(\sqrt{2}y)+K_{18}\sin(\sqrt{2}y)\big),$
	$G''(z^*)+3G(z^*)=0.$	$v(x,y,z)=e^{yz}\big(K_{19}\cos(\sqrt{3}y)+K_{20}\sin(\sqrt{3}y)\big).$

3.6 System 6 or (2.91)

Reduction by $X_3 + X_7$ yields the system

$$w^1_{z_1z_1} + w^1_{z_2z_2} = \alpha w^1 + w^2, \tag{3.60a}$$
$$w^2_{z_1z_1} + w^2_{z_2z_2} = w^2, \tag{3.60b}$$

which has symmetries

$$\begin{aligned} Y_1 = \partial_{z_1}, Y_2 = \partial_{z_2},\ Y_3 = z_2\partial_{z_1} - z_1\partial_{z_2}, Y_4 = w^1\partial_{w^1} + w^2\partial_{w^2}, \\ Y_5 = (w^1(\alpha-2) + w^2)\partial_{w^1}, Y_a = a(z_1,z_2)\partial_{w^1},\ Y_b = b(z_1,z_2)\partial_{w^2}. \end{aligned} \tag{3.61}$$

Table 3.12 presents invariant solutions.

Table 3.12: Invariant solutions of Lie algebra (3.61)

Subalgebra	Reduced System	Invariant Solution
Y_1	$F''(z^*) - \alpha F(z^*) = G(z^*),$ $G''(z^*) - G(z^*) = 0.$	$u(x,y,z) = e^x\left(\frac{K_1}{1-\alpha}e^z + \frac{K_2}{1-\alpha}e^{-z} + K_3 e^{\sqrt{\alpha}z} + K_4 e^{-\sqrt{\alpha}z}\right),$ $v(x,y,z) = e^x\left(K_3 e^z + K_4 e^{-z}\right).$
Y_2	$F''(z^*) - \alpha F(z^*) = G(z^*),$ $G''(z^*) - G(z^*) = 0.$	$u(x,y,z) = e^x\left(\frac{K_5}{1-\alpha}e^y + \frac{K_6}{1-\alpha}e^{-y} + K_7 e^{\sqrt{\alpha}y} + K_8 e^{-\sqrt{\alpha}y}\right),$ $v(x,y,z) = e^x\left(K_5 e^y + K_6 e^{-y}\right)$
$Y_1 + Y_2$	$2F''(z^*) + (1-\alpha)F(z^*) = G(z^*),$ $2G''(z^*) - G(z^*) = 0.$	$u(x,y,z) = e^x\left(K_9 \exp\left(\sqrt{\frac{\alpha-2}{2}}(y-z)\right) + K_{10}\exp\left(-\sqrt{\frac{\alpha-2}{2}}(y-z)\right) + \frac{K_{11}}{2-\alpha}e^{\sqrt{\frac{1}{2}}(y-z)} + \frac{K_{12}}{2-\alpha}e^{-\sqrt{\frac{1}{2}}(y-z)}\right),$ $v(x,y,z) = e^x\left(K_{11}e^{\sqrt{\frac{1}{2}}(y-z)} + K_{12}e^{-\sqrt{\frac{1}{2}}(y-z)}\right).$
$Y_1 + Y_4$	$F''(z^*) + (1-\alpha)F(z^*) = G(z^*),$ $G''(z^*) = 0.$	$u(x,y,z) = e^{xy}\left(\frac{K_{15}}{1-\alpha}z + \frac{K_{16}}{1-\alpha} + K_{13}e^{\sqrt{1-\alpha}z} + K_{14}e^{-\sqrt{1-\alpha}z}\right),$ $v(x,y,z) = e^{xy}\left(K_{15}z + K_{16}\right)$
$Y_2 + Y_4$	$F''(z^*) + (1-\alpha)F(z^*) = G(z^*),$ $G''(z^*) = 0.$	$u(x,y,z) = e^{xz}\left(\frac{K_{19}}{1-\alpha}y + \frac{K_{20}}{1-\alpha} + K_{17}e^{\sqrt{1-\alpha}y} + K_{18}e^{-\sqrt{1-\alpha}y}\right),$ $v(x,y,z) = e^{xz}\left(K_{19}y + K_{20}\right).$

3.7 System 7 or (2.109)

Using the symmetry $X_2 = \partial_y$ to reduce system (2.109) leads to a system of PDEs

$$w^1_{z_1z_1} + w^1_{z_2z_2} = -\alpha w^1 - w^2 \tag{3.62a}$$

$$w^2_{z_1z_1} + w^2_{z_2z_2} = -w^2. \tag{3.62b}$$

The symmetries of system (3.65) read

$$\begin{gathered} Y_1 = \partial_{z_1}, Y_2 = \partial_{z_2},\ Y_3 = z_2\partial_{z_1} - z_1\partial_{z_2}, Y_4 = w^1\partial_{w^1} + w^2\partial_{w^2}, \\ Y_5 = (w^1(\alpha-2) + w^2)\partial_{w^1}, Y_a = a(z_1,z_2)\partial_{w^1},\ Y_b = b(z_1,z_2)\partial_{w^2}. \end{gathered} \tag{3.63}$$

Table 3.13 gives a list of invariant solutions associated with the linear combinations of Lie algebra (3.63).

Table 3.13: Invariant solutions of Lie algebra (3.63)

Subalgebra	Reduced System	Invariant Solution
Y_1	$F''(z^*)+\alpha F(z^*)=-G(z^*),$ $G''(z^*)+G(z^*)=0.$	$u(x,y,z)=\frac{K_1}{1-\alpha}\cos(z)+\frac{K_2}{1-\alpha}\sin(z)+K_3\cos\left(\sqrt{\alpha}z\right)+K_4\sin\left(\sqrt{\alpha}z\right),$ $v(x,y,z)=K_1\cos(z)+K_2\sin(z).$
Y_2	$F''(z^*)+\alpha F(z^*)=-G(z^*),$ $G''(z^*)+G(z^*)=0.$	$u(x,y,z)=\frac{K_5}{1-\alpha}\cos(x)+\frac{K_6}{1-\alpha}\sin(x)+K_7\cos\left(\sqrt{\alpha}x\right)+K_8\sin\left(\sqrt{\alpha}x\right),$ $v(x,y,z)=K_5\cos(x)+K_6\sin(x)$
Y_1+Y_4	$F''(z^*)+(\alpha+1)F(z^*)=-G(z^*),$ $G''(z^*)+2G(z^*)=0.$	$u(x,y,z)=e^x\left(K_{11}\cos\left(\sqrt{\alpha-1}z\right)+K_{12}\sin\left(\sqrt{\alpha-1}z\right)-\frac{K_9}{\alpha}\cos\left(\sqrt{2}z\right)-\frac{K_{10}}{\alpha}\sin\left(\sqrt{2}z\right)\right),$ $v(x,y,z)=e^x\left(K_9\cos\left(\sqrt{2}z\right)+K_{10}\sin\left(\sqrt{2}z\right)\right).$
Y_2+Y_4	$F''(z^*)+(\alpha+1)F(z^*)=-G(z^*),$ $G''(z^*)=0.$	$u(x,y,z)=e^z\left(K_{15}\cos\left(\sqrt{\alpha-1}x\right)+K_{15}\sin\left(\sqrt{\alpha-1}x\right)-\frac{K_{13}}{\alpha}\cos\left(\sqrt{2}x\right)-\frac{K_{14}}{\alpha}\sin\left(\sqrt{2}x\right)\right),$ $v(x,y,z)=e^z\left(K_{13}\cos\left(\sqrt{2}x\right)+K_{14}\sin\left(\sqrt{2}x\right)\right).$

3.8 System 8 or (2.110)

Invariance under X_4 leads to the invariants

$$z_1 = x^2 + y^2,\ z_2 = z,\ w^1 = u,\ w^2 = v. \tag{3.64}$$

The invariants reduce system (2.110) to a system of PDEs

$$4z_1 w^1_{z_1 z_1} + 4w^1_{z_1} + w^1_{z_2 z_2} = \alpha w^1 + \beta w^2, \tag{3.65a}$$
$$4z_1 w^2_{z_1 z_1} + 4w^2_{z_1} + w^2_{z_2 z_2} = -\beta w^1 + \alpha w^2, \tag{3.65b}$$

which has symmetries

$$\begin{aligned} &Y_1 = \partial_{z_2}, \quad Y_2 = w^1 \partial_{w^1} + w^2 \partial_{w^2}, \quad Y_3 = w^2 \partial_{w^1} - w^1 \partial_{w^2}, \\ &Y_A = \frac{1}{\beta}\left(\alpha A - 4A_{z_1} - 4z_1 A_{z_1 z_1} - A_{z_2 z_2}\right)\partial_{w^1} + A(z_1, z_2)\partial_{w^2}, \end{aligned} \tag{3.66}$$

where $A(z_1, z_2)$ satisfies the equation

$$\begin{aligned} \left(\alpha^2 + \beta^2\right) A - 8\alpha A_{z_1} + 8\left(3 - \alpha z_1\right) A_{z_1 z_1} - 2\alpha A_{z_2 z_2} + 64 z_1 A_{z_1 z_1 z_1} + 8 A_{z_1 z_2 z_2} + 16 z_1^2 A_{z_1 z_1 z_1 z_1} \\ + 8 z_1 A_{z_1 z_1 z_2 z_2} + A_{z_2 z_2 z_2 z_2} = 0. \end{aligned}$$

To further reduce system (3.65) to a system of ODEs, we use linear combinations of symmetries (3.66). Only combinations that reduce the system are shown.

Reduction by Y_1:

$$\alpha F + \beta G - 4\left(F' + z^* F''\right) = 0, \tag{3.67a}$$
$$\beta F - \alpha G + 4\left(G' + z^* G''\right) = 0. \tag{3.67b}$$

Reduction by $Y_1 + Y_2$:

$$(\alpha - 1)F + \beta G - 4\left(F' + z^* F''\right) = 0, \tag{3.68a}$$
$$\beta F - (\alpha - 1)G + 4\left(G' + z^* G''\right) = 0. \tag{3.68b}$$

Reduction by $Y_1 + Y_3$:

$$\begin{aligned} \left[\beta \cos z - (1+\alpha)\sin z\right] F + \left[(1+\alpha)\cos z + \beta \sin z\right] G + 4\left(F' + z^* F''\right)\sin z \\ -4\left(G' + z^* G''\right)\cos z = 0, \end{aligned} \tag{3.69a}$$
$$\begin{aligned} \left[(\alpha+1)\sin z - \beta \cos z\right] G + \left[(\alpha+1)\cos z + \beta \sin z\right] F - 4\left(F' + z^* F''\right)\cos z \\ -4\left(G' + z^* G''\right)\sin z = 0. \end{aligned} \tag{3.69b}$$

Here "$'$" in systems (3.67),(3.68),(3.69) indicates derivative with respect to z^*. The reduced ODEs systems cannot be solved analytically by the elementary techniques or the Mathematica DSOLVE command.

Invariance under $X_2 + X_7$ gives invariants

$$z_1 = x, \quad z_2 = z, \quad u = e^y w^1(z_1, z_2), \quad v = e^y w^2(z_1, z_2). \tag{3.70}$$

Invariants (3.70) reduce system (2.110) to the system

$$w^1_{z_1z_1} + w^1_{z_2z_2} + (1-\alpha)\,w^1 - \beta w^2 = 0, \tag{3.71a}$$
$$w^2_{z_1z_1} + w^2_{z_2z_2} + (1-\alpha)\,w^2 + \beta w^1 = 0, \tag{3.71b}$$

which has symmetries

$$\begin{gathered} Y_1 = \partial_{z_1}, \quad Y_2 = \partial_{z_2}, \quad Y_3 = z_2\partial_{z_1} - z_1\partial_{z_2}, \quad Y_4 = w^1\partial_{w^1} + w^2\partial_{w^2}, \\ Y_5 = w^2\partial_{w^1} - w^1\partial_{w^2}, \quad Y_A = \frac{1}{\beta}\left[(\alpha-1)A - A_{z_1z_1} - A_{z_2z_2}\right]\partial_{w^1} + A(z_1,z_2)\partial_{w^2}, \end{gathered} \tag{3.72}$$

where $A(z_1,z_2)$ satisfies the equation

$$\left(2\alpha - \alpha^2 - \beta^2 - 1\right)A + 2(\alpha-1)\left(A_{z_1z_1} + A_{z_2z_2}\right) - A_{z_1z_1z_1z_1} - 2A_{z_1z_1z_2z_2} + A_{z_2z_2z_2z_2} = 0.$$

Reduction by combinations $Y_1,\ Y_2,\ Y_1+Y_2,\ Y_1+Y_4,\ Y_2+Y_4,\ Y_2+Y_5,\ Y_1+Y_2+Y_4$ lead to complex solutions.

Reduction by $Y_3,\ Y_3+Y_4,\ Y_3+Y_5$ lead to ODEs systems which cannot be solved analytically by the elementary techniques or the Mathematica DSOLVE command.

Reduction by Y_1+Y_5 leads to a Mathematica generated solution which spans 185 pages when saved as pdf file, as a result the solution is omitted.

Note: Invariance under X_5, X_6, X_1+X_8, X_2+X_7, X_2+X_8, $X_1+X_2+X_3+X_7$ is also possible, this was observed by trial and error. This list is not exhaustive because the optimal system of subalgebras was not constructed. The optimal system of subalgebras for all the original elliptic systems will be a subject of future work.

Chapter 4

Conservation Laws of Elliptic Systems

Conservation laws and symmetries have always been far-reaching in mathematics and science as they are applicable in the formulation and investigation of various models. Amongst others, they are used for proving global existence theorems [11, 4], in problems of stability [6, 25] and in fluid mechanics [8]. If a system does not interact with the surrounding at all, then it means certain mechanical properties of the system cannot change. These quantities are said to be conserved and the conservation laws involved are fundamental principles of mechanics. Such conserved quantities include energy, momentum, and angular momentum.

There are many methods used for constructing conservation laws for DEs including the direct method [2, 12, 15, 8], the symmetry/adjoint symmetry pair method [1] and the Noether's approach among others. A systematic way of constructing the conservation laws of a system of DEs that admits a variational principle is via Noether's theorem. Many references including [12, 14, 15, 20, 21] explain this method. The method used for calculation of conservation laws in this dissertation is via the Noether's approach.

4.1 Preliminaries

Most of the theoretical concepts and notations presented in this section are adopted from [13].

Definition 4.1.1 A locally analytic function of a finite number of independent variables, dependent variables and derivatives of the dependent variables is called a differential function. The vector space of all differential functions of finite order is denoted by $\mathcal{F}$.

Definition 4.1.2 A differential operator of the form

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{4.1}$$

where $\xi^i, \eta^\alpha \in \mathcal{F}$ and

$$\begin{aligned} \zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ &\;\vdots \\ \zeta_{i_1 \cdots i_s}^\alpha &= D_{i_1} \cdots D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \cdots i_s}^\alpha, \, s > 1, \end{aligned} \tag{4.2}$$

is called a Lie-Backlund operator. In (4.2), W^α is the Lie characteristic function given by

$$W^\alpha = n^\alpha - \xi^i u_j^\alpha. \tag{4.3}$$

Definition 4.1.3 A function $L = L(x, u^\alpha, u_i^\alpha)$ is a first-order Lagrangian of the second-order system of DEs if it satisfies the Euler-Lagrange system

$$\frac{\delta L}{\delta u^\alpha} = 0. \tag{4.4}$$

Definition 4.1.4 The operator in (4.4), that is

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1}(-1)^s D_{i_1}...D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1...i_s}}, \alpha = 1,2,...,m \tag{4.5}$$

is called the Euler-Lagrange operator.

Definition 4.1.5 A Lie-Backlund operator X is said to be a Noether (point) symmetry corresponding to a Lagrangian $L \in \mathcal{F}$ of the Euler-Lagrange system if there exists a vector

$$B^i = (B^1, B^2, ..., B^n) \in \mathcal{F},$$

such that

$$X(L) + L\text{Div}(\xi^i) = \text{Div}(B^i). \tag{4.6}$$

If $B^i = 0$ (for $i = 1,2,...,n$), then X is a strict Noether symmetry corresponding to a Lagrangian L.

Theorem 4.1.6 For any Noether symmetry X corresponding to a Lagrangian L there corresponds a vector $T^i = (T^1, T^2, ..., T^n)$, $T^i \in \mathcal{F}$ defined by

$$T^i = N^i(L) - B^i, i = 1,2,...,n \tag{4.7}$$

which is a conserved vector of an Euler-Lagrange equation (or system) $\frac{\delta L}{\delta u^\alpha} = 0$, and the Noether operator associated with X is

$$N^i = \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} + \sum_{s\geq 1} D_{i_1}...D_{i_s}(W^\alpha)\frac{\delta}{\delta u^\alpha_{ii_1...i_s}}. \tag{4.8}$$

Theorem 4.1.7 For any Lie-Backlund symmetry X and the components of the conserved vectors T^i, then

$$X(T^i) + T^i D_k(\xi^k) - T^k D_k(\xi^i) = N^i(D_k(B^k)) + B^k D_k(\xi^i) - B^i D_k(\xi^k) - X(B^i). \tag{4.9}$$

If (4.9) is satisfied, then X is said to be associated with the Noether conserved vector T^i. A vector T^i is said to be conserved if it satisfies $D_i T^i = 0$ along the solutions of the given PDE or a system of PDEs, i.e.,

$$D_i T^i\big|_{E^\alpha} = 0, \text{ where } E^\alpha \text{ is the system of DEs.} \tag{4.10}$$

Equation (4.10) is a conservation law. The Noether's theorem guarantees a conservation law for each Noether symmetry associated with a Lagrangian. In the case in which there is no prior knowledge of a Lagrangian, (i.e., problems which are not variational) conservation laws can be obtained using other methods [8].

4.2 Conserved vectors

Consider system (2.34) discussed in Chapter 2 (referred to as System 3). We suppose that the Lagragian L exists such that (2.34) can be expressed in Euler-Lagrange form

$$f(u_{xx}+u_{yy}+u_{zz})=\frac{\delta L}{\delta u}, \tag{4.11a}$$
$$g(v_{xx}+v_{yy}+v_{zz})=\frac{\delta L}{\delta v}, \tag{4.11b}$$

where $L=L(x,y,z,u_x,u_y,u_z)$. Thus

$$f(u_{xx}+u_{yy}+u_{zz})=\frac{\partial L}{\partial u}-D_x\frac{\partial L}{\partial u_x}-D_y\frac{\partial L}{\partial u_y}-D_z\frac{\partial L}{\partial u_z}, \tag{4.12a}$$
$$g(v_{xx}+v_{yy}+v_{zz})=\frac{\partial L}{\partial v}-D_x\frac{\partial L}{\partial v_x}-D_y\frac{\partial L}{\partial v_y}-D_z\frac{\partial L}{\partial v_z}, \tag{4.12b}$$

where

$$D_x=\frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xy}\frac{\partial}{\partial u_y}+u_{xz}\frac{\partial}{\partial u_z}+\cdots, \tag{4.13a}$$
$$D_y=\frac{\partial}{\partial x}+u_y\frac{\partial}{\partial u}+u_{xy}\frac{\partial}{\partial u_x}+u_{yy}\frac{\partial}{\partial u_y}+u_{yz}\frac{\partial}{\partial u_z}+\cdots, \tag{4.13b}$$
$$D_z=\frac{\partial}{\partial z}+u_z\frac{\partial}{\partial u}+u_{xz}\frac{\partial}{\partial u_x}+u_{yz}\frac{\partial}{\partial u_y}+u_{zz}\frac{\partial}{\partial u_z}+\cdots. \tag{4.13c}$$

Expanding equations (4.12) and separating by second order derivatives of u and v yields the following determining equations

$$L_{u_xu_x}=f, \tag{4.14a}$$
$$L_{u_xv_x}=0, \tag{4.14b}$$
$$L_{v_xv_x}=0, \tag{4.14c}$$
$$L_{u_xu_y}=0, \tag{4.14d}$$
$$L_{u_yv_x}+L_{u_xv_y}=0, \tag{4.14e}$$
$$L_{v_xv_y}=0, \tag{4.14f}$$
$$L_{u_xu_z}=0, \tag{4.14g}$$
$$L_{u_zv_x}+L_{u_xv_z}=0, \tag{4.14h}$$
$$L_{v_xv_z}=0, \tag{4.14i}$$
$$L_{u_yu_y}=f, \tag{4.14j}$$
$$L_{u_yv_y}=0, \tag{4.14k}$$
$$L_{v_yv_y}=g, \tag{4.14l}$$
$$L_{u_yv_z}=0, \tag{4.14m}$$
$$L_{u_zv_y}+L_{u_yv_z}=0, \tag{4.14n}$$
$$L_{v_yv_z}=0, \tag{4.14o}$$
$$L_{u_zu_z}=f, \tag{4.14p}$$
$$L_{u_zv_z}=0, \tag{4.14q}$$

$$L_{v_z v_z} = g, \tag{4.14r}$$

$$L_v - \Big(L_{xv_x} + u_x L_{uv_x} + v_x L_{vv_x} + L_{yv_y} + u_y L_{uv_y} + v_y L_{vv_y} + L_{zv_z} + u_z L_{uv_z} + v_z L_{vv_z}\Big)$$
$$+ L_u - \Big(L_{xu_x} + u_x L_{uu_x} + v_x L_{vu_x} + L_{yu_y} + u_y L_{uu_y} + v_y L_{vu_y} + L_{zu_z} + u_z L_{uu_z} + v_z L_{vu_z}\Big) = 0. \tag{4.14s}$$

From equations (4.14) we find the following general Lagrangian

$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + u_x\big(v_y v_z a^1 + v_y a^2 + v_z a^3 + a^4\big)$$
$$u_y\big(v_x v_z b^1 + v_x b^2 + v_z b^3 + b^4\big) + u_z\big(v_x v_y c^1 + v_x c^2 + v_y c^3 + c^4\big) + v_x d^1 + v_y d^2 + v_z d^3 + d^4, \tag{4.15}$$

where arbitrary functions a^i, b^i, c^i, d^i; $i = 1,2,3,4$ of x, y, z, satisfy the following equations

$$a^1 + b^1 = 0,$$
$$a^2 + b^2 = 0,$$
$$a^1 + c^1 = 0,$$
$$a^3 + c^2 = 0,$$
$$c^1 + b^1 = 0,$$
$$c^3 + b^3 = 0.$$

To find a particular Lagrangian, from (4.15), we let all except one of a^i, b^i, c^i and d^i be equal to zero. Below is a list of possible Lagrangians

$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g, \tag{4.16a}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + u_x a^4, \tag{4.16b}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + u_y b^4, \tag{4.16c}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + u_z c^4, \tag{4.16d}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + v_x d^1, \tag{4.16e}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + v_y d^2, \tag{4.16f}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + v_z d^3, \tag{4.16g}$$
$$L = \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_z^2}{2}\Big) f + \Big(\frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big) g + d^4. \tag{4.16h}$$

4.3 Noether symmetries

According to Noether's theorem, each of the Lagrangians in (4.16) gives rise to the corresponding Noether symmetries and hence for each Lagrangian, conserved vectors can be obtained. We will consider the Lagrangian (4.16a) when arbitrary functions $f = g = 1$. Using (4.6), we have

$$X^{[1]}L + L\text{Div}(\xi^1, \xi^2, \xi^3) = \text{Div}(B^1, B^2, B^3), \tag{4.17}$$

where $X^{[1]}$ is given by (1.18). Substituting (4.16a) into (4.17) yields

$$D_xB^1 + D_yB^2 + D_zB^3 = \Big(u_x\zeta^1_x + u_y\zeta^1_y + u_z\zeta^1_z\Big) + \Big(v_x\zeta^2_x + v_y\zeta^2_y + v_z\zeta^2_z\Big) + \Big(\frac{u_x^2}{2} + \frac{u_y^2}{2} + \frac{u_x^2}{2} + \frac{v_x^2}{2} + \frac{v_y^2}{2} + \frac{v_z^2}{2}\Big)\Big(D_x\xi^1 + D_y\xi^2 + D_z\xi^3\Big). \tag{4.18}$$

Expanding (4.18) and separating by powers and products of derivatives of u and v yields the determining equations

$$\eta^1_v + \eta^2_u = 0, \tag{4.19a}$$
$$\eta^1_u + \frac{1}{2}\Big(-\xi^1_x + \xi^2_y + \xi^3_z\Big) = 0, \tag{4.19b}$$
$$\eta^1_u + \frac{1}{2}\Big(\xi^1_x - \xi^2_y + \xi^3_z\Big) = 0, \tag{4.19c}$$
$$\eta^1_u + \frac{1}{2}\Big(\xi^1_x + \xi^2_y - \xi^3_z\Big) = 0, \tag{4.19d}$$
$$\eta^2_v + \frac{1}{2}\Big(-\xi^1_x + \xi^2_y + \xi^3_z\Big) = 0, \tag{4.19e}$$
$$\eta^2_v + \frac{1}{2}\Big(\xi^1_x - \xi^2_y + \xi^3_z\Big) = 0, \tag{4.19f}$$
$$\eta^2_v + \frac{1}{2}\Big(\xi^1_x + \xi^2_y - \xi^3_z\Big) = 0 \tag{4.19g}$$
$$\eta^1_x - B^1_u = 0, \tag{4.19h}$$
$$\eta^1_y - B^2_u = 0, \tag{4.19i}$$
$$\eta^1_z - B^3_u = 0, \tag{4.19j}$$
$$\eta^2_x - B^1_v = 0, \tag{4.19k}$$
$$\eta^2_y - B^2_v = 0, \tag{4.19l}$$
$$\eta^2_z - B^3_v = 0, \tag{4.19m}$$
$$\xi^1_y + \xi^2_x = 0, \tag{4.19n}$$
$$\xi^1_z + \xi^3_x = 0, \tag{4.19o}$$
$$\xi^2_z + \xi^3_y = 0, \tag{4.19p}$$
$$\xi^1_u = 0, \tag{4.19q}$$
$$\xi^1_v = 0, \tag{4.19r}$$
$$\xi^2_u = 0, \tag{4.19s}$$
$$\xi^2_v = 0, \tag{4.19t}$$
$$\xi^3_u = 0, \tag{4.19u}$$
$$\xi^3_v = 0, \tag{4.19v}$$
$$B^1_x + B^2_y + B^3_z = 0. \tag{4.19w}$$

From equations (4.19) the following solutions are obtained

$$\xi^1 = p(x, y, z), \tag{4.20a}$$

$$\xi^2 = q(x,y,z), \tag{4.20b}$$
$$\xi^3 = r(x,y,z), \tag{4.20c}$$
$$\eta^1 = -u\frac{p_x}{2} + vK_1 + d(x,y,z), \tag{4.20d}$$
$$\eta^2 = -v\frac{p_x}{2} - vK_1 + c(x,y,z), \tag{4.20e}$$

and

$$B^1 = -\Big(\frac{u^2+v^2}{4}\Big)p_{xx} + ud_x + vc_x + \alpha(x,y,z), \tag{4.21a}$$
$$B^2 = -\Big(\frac{u^2+v^2}{4}\Big)p_{xy} + ud_y + vc_y + \beta(x,y,z), \tag{4.21b}$$
$$B^3 = -\Big(\frac{u^2+v^2}{4}\Big)p_{xz} + ud_z + vc_z + \gamma(x,y,z), \tag{4.21c}$$

where arbitrary functions of x, y, z satisfy the following

$$p_y + q_x = 0, \tag{4.22a}$$
$$p_z + r_x = 0, \tag{4.22b}$$
$$q_z + r_y = 0, \tag{4.22c}$$
$$p_{xx} + p_{yy} + p_{zz} = F(y,z), \tag{4.22d}$$
$$d_{xx} + d_{yy} + d_{zz} = 0, \tag{4.22e}$$
$$\alpha_x + \beta_y + \gamma_z = 0. \tag{4.22f}$$

From (4.20) we get the following Noether point symmetries

$$X_1 = v\partial_u - u\partial_v, \tag{4.23a}$$
$$X_{p_x} = up_x\partial_u + vp_x\partial_v, \tag{4.23b}$$
$$X_p = p(x,y,z)\partial_x, \tag{4.23c}$$
$$X_q = q(x,y,z)\partial_y, \tag{4.23d}$$
$$X_r = r(x,y,z)\partial_z, \tag{4.23e}$$
$$X_d = d(x,y,z)\partial_u, \tag{4.23f}$$
$$X_c = c(x,y,z)\partial_v. \tag{4.23g}$$

To find the conserved vectors, we use the formula

$$T^i = \xi^i L + W^1\frac{\partial L}{\partial u_i} + W^2\frac{\partial L}{\partial v_i} - B^i \tag{4.24}$$

where

$$W^1 = \eta^1 - u_x\xi^1 - u_y\xi^2 - u_z\xi^3,$$
$$W^2 = \eta^2 - v_x\xi^1 - v_y\xi^2 - v_z\xi^3.$$

Next we find conserved vectors associated with finite symmetries.

For $X_1 = v\partial_u - u\partial_v$, we have: $\xi^1 = \xi^2 = \xi^3 = 0$, $\eta^1 = v$ and $\eta^2 = -u$. The corresponding conserved vectors are

$$T^1 = vu_x - uv_x - ud_x - vc_x - \alpha, \tag{4.26a}$$
$$T^2 = vu_y - uv_y - ud_y - vc_y - \beta, \tag{4.26b}$$
$$T^3 = vu_z - uv_z - ud_z - vc_z - \gamma. \tag{4.26c}$$

The rest of Lagrangians given in (4.16) have the same Noether symmetries given by (4.23). What differs for each Lagrangian are the values of $B^i s$. Table 4.1 gives the B^i for each Lagrangian together with the corresponding conserved vectors for the finite symmetry X_1. Through calculations, it was established that the Lagrangian (4.16h) possesses the conserved vectors (4.26).

Table 4.1: List of conserved vectors

Lagrangian	B^i	Conserved vectors
	$B^1 = -\left(\frac{u^2+v^2}{4}\right)p_{xx} + ud_x + vc_x - \frac{3}{2}ua^4p_x + va^4K_1 + u(pa_x^4 + qa_y^4 + ra_z^4), +\alpha(x,y,z),$	$T^1 = v(u_x + a^4) - uv_x - \alpha -\left(ud_x + vc_x + va^4K_1 + u(pa_x^4 + qa_y^4 + ra_z^4)\right),$
(4.16b)	$B^2 = -\left(\frac{u^2+v^2}{4}\right)p_{xy} + ud_y + vc_y - ua^4q_y,$	$T^2 = vu_y - uv_y - \left(ud_y + vc_y + \beta\right),$
	$B^3 = -\left(\frac{u^2+v^2}{4}\right)p_{xz} + ud_z + vc_z - ua^4r_z + \gamma(x,y,z).$	$T^3 = vu_z - uv_z - \left(ud_z + vc_z + \gamma\right).$
	$B^1 = -\left(\frac{u^2+v^2}{4}\right)p_{xx} + ud_x + vc_x - ub^4q_y + \alpha(x,y,z),$	$T^1 = vu_x - uv_x - \left(ud_x + vc_x + \alpha\right),$
(4.16c)	$B^2 = -\left(\frac{u^2+v^2}{4}\right)p_{xy} + ud_y + vc_y - ub^4\frac{p_x}{2} - ub^4q_y + vb^4K_1 + u(pa_x^4 + qa_y^4 + ra_z^4) +\beta(x,y,z),$	$T^2 = v(u_y + b^4) - uv_y - \beta -\left(ud_y + vc_y + vb^4K_1 + u(pa_x^4 + qa_y^4 + ra_z^4)\right),$
	$B^3 = -\left(\frac{u^2+v^2}{4}\right)p_{xz} + ud_z + vc_z - ub^4r_z + \gamma(x,y,z).$	$T^3 = vu_z - uv_z - \left(ud_z + vc_z + \gamma\right).$
	$B^1 = -\left(\frac{u^2+v^2}{4}\right)p_{xx} + ud_x + vc_x - uc^4p_z + \alpha(x,y,z),$	$T^1 = vu_x - uv_x - \left(ud_x + vc_x + \alpha\right),$
(4.16d)	$B^2 = -\left(\frac{u^2+v^2}{4}\right)p_{xy} + ud_y + vc_y - uc^4q_y + \beta(x,y,z).$	$T^2 = vu_y - uv_y - \left(ud_y + vc_y + \beta\right),$
	$B^3 = -\left(\frac{u^2+v^2}{4}\right)p_{xz} + ud_z + vc_z + vc^4K_1 - uc^4\frac{p_x}{2} - uc^4r_z + u(pa_x^4 + qa_y^4 + ra_z^4) +\gamma(x,y,z).$	$T^3 = v(u_z + c^4) - uv_z - \gamma -\left(ud_z + vc_z + vc^4K_1 + u(pa_x^4 + qa_y^4 + ra_z^4)\right).$

Lagrangian	B^i	Conserved vectors
	$B^1=-\left(\frac{u^2+v^2}{4}\right)p_{xx}+ud_x+vc_x-\frac{3}{2}vd^1p_x-ud^1K_1+v(pd^1_x+qd^1_y+rd^1_z)$ $+\alpha(x,y,z)$,	$T^1=v(u_x+d^1)-uv_x-\alpha$ $-\big(ud_x+vc_x-ud^1K_1+v(pa^4_x+qa^4_y+ra^4_z)\big)$,
(4.16e)	$B^2=-\left(\frac{u^2+v^2}{4}\right)p_{xy}+ud_y+vc_y-vd^1q_x+\beta(x,y,z)$,	$T^2=vu_y-uv_y-\big(ud_y+vc_y+\beta\big)$,
	$B^3=-\left(\frac{u^2+v^2}{4}\right)p_{xz}+ud_z+vc_z-vd^1r_x+\gamma(x,y,z)$.	$T^3=vu_z-uv_z-\big(ud_z+vc_z+\gamma\big)$.
	$B^1=-\left(\frac{u^2+v^2}{4}\right)p_{xx}+ud_x+vc_x-vd^2p_y+\alpha(x,y,z)$,	$T^1=vu_x-uv_x-\big(ud_x+vc_x+\alpha\big)$,
(4.16f)	$B^2=-\left(\frac{u^2+v^2}{4}\right)p_{xy}+ud_y+vc_y+vd^2\frac{p_x}{2}-vd^2q_y-ud^2K_1+v(pa^4_x+qa^4_y+ra^4_z)$ $+\beta(x,y,z)$,	$T^2=v(u_y+d^2)-uv_y-\beta$ $-\big(ud_y+vc_y-ud^2K_1+u(pa^4_x+qa^4_y+ra^4_z)\big)$,
	$B^3=-\left(\frac{u^2+v^2}{4}\right)p_{xz}+ud_z+vc_z-vd^2r_y+\gamma(x,y,z)$.	$T^3=vu_z-uv_z-\big(ud_z+vc_z+\gamma\big)$.
	$B^1=-\left(\frac{u^2+v^2}{4}\right)p_{xx}+ud_x+vc_x-vd^3p_z+\alpha(x,y,z)$,	$T^1=vu_x-uv_x-\big(ud_x+vc_x+\alpha\big)$,
(4.16g)	$B^2=-\left(\frac{u^2+v^2}{4}\right)p_{xy}+ud_y+vc_y-vd^3q_z+\beta(x,y,z)$.	$T^2=vu_y-uv_y-\big(ud_y+vc_y+\beta\big)$,
	$B^3=-\left(\frac{u^2+v^2}{4}\right)p_{xz}+ud_z+vc_z-vd^3\frac{p_x}{2}-vd^3r_z+v(pa^4_x+qa^4_y+ra^4_z)$ $+\gamma(x,y,z)$.	$T^3=v(u_z+d^3)-uv_z-\gamma$ $-\big(ud_z+vc_z+v(pa^4_x+qa^4_y+ra^4_z)\big)$.

Chapter 5

Conclusion

In this work, systems of elliptic type in three independent variables were investigated using symmetry methods; eight systems of equations were considered. The Lie point symmetries for each system were obtained. Through calculations, it was discovered that the following pairs of systems admitted the same symmetry Lie algebra: systems 1 & 2, 4 & 5 and 6 & 7. All the systems possess an infinite-dimensional Lie algebra having the common symmetries $X_1, X_2, ..., X_6$ together with the infinite symmetries. Some finite symmetries and their linear combinations were used to reduce the original system to an invariant system with two new independent variables whose symmetries were also obtained. The optimal system of one-dimensional subalgebras for the new symmetries were used for reduction to a system of ODEs. Some symmetry reductions were performed for all the systems and various types of invariant solutions were derived. Since not all the finite symmetries were used to reduce a particular original elliptic system, the future work to be presented elsewhere will involve construction of optimal system of two-dimensional subalgebras for that original system.

The last part of this work involves the derivation of conservation laws. Through calculations, it was proven that all elliptic systems led to similar Lagrangians and hence similar conserved vectors were obtained. Therefore only calculations for one system (System 3) are presented.

Bibliography

[1] Stephen C. Anco , George Bluman W. and Alexei F. Cheviakov. *Applications of symmetry methods to partial differential equations*, volume 168. Springer, 2010.

[2] Stephen C. Anco and George Bluman. Direct construction of conservation laws from field equations. *Physical Review Letters*, 78(15):2869, 1997.

[3] George W. Bluman and Sukeyuki Kumei. *Symmetries and differential equations*. Springer Science & Business Media, 2013.

[4] Takahashi Daisuke, Sachs Jeffrey R. and Junkichi Satsuma. Properties of the magma and modified magma equations. *Journal of the Physical Society of Japan*, 59(6):1941–1953, 1990.

[5] Jose M. Díaz. Yet another Lie (YaLie) Mathematica package for Lie symmetries, Version 2.1, Departmento de Matematicas, Universidad de Cadiz. *library.wolfram.com/infocenter/MathSource/4231*.

[6] Francis A.L. Dullien. *Porous media: fluid transport and pore structure*. Academic press, 2012.

[7] Louis Dupaigne. *Stable solutions of elliptic partial differential equations*. CRC press, 2011.

[8] Mahomed M. Fazal, David P. Mason and Rahila Naz. Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics. *Applied Mathematics and Computation*, 205(1):212–230, 2008.

[9] Mostafa Fazly and Henrik Shahgholian. Monotonicity formulas for coupled elliptic gradient systems with applications. *Advances in Nonlinear Analysis*, 9(1):479–495, 2019.

[10] Petter L. Hans and Anders Logg. Fundamentals: Solving the poisson equation. In *Solving PDEs in Python*, pages 11–35. Springer, 2016.

[11] Nail H. Ibragimov. A new conservation theorem. *Journal of Mathematical Analysis and Applications*, 333(1):311–328, 2007.

[12] Nail H. Ibragimov. *Selected works*. ALGA publications, Blekinge Institute of Technology, 2009.

[13] Abdul H. Kara. Lecture notes on Symmetries and Conservation Laws, University of the Witwatersrand, Johannesburg, 2002.

[14] Abdul H. Kara and Fazal M. Mahomed. Relationship between symmetries and conservation laws. *International Journal of Theoretical Physics*, 39(1):23–40, 2000.

[15] Abdul H. Kara and Fazal M. Mahomed. A basis of conservation laws for partial differential equations. *Journal of Nonlinear Mathematical Physics*, 9(2):60–72, 2002.

[16] Boccardo Lucio, Orsina Luigi and Alessio Porretta. Strongly coupled elliptic equations related to mean-field games systems. *Journal of Differential Equations*, 261(3):1796–1834, 2016.

[17] Bernd Markert. *Weak or strong: on coupled problems in continuum mechanics*. Universität Stuttgart, 2010.

[18] Enzo Mitidieri and Guido Sweers. Weakly coupled elliptic systems and positivity. *Mathematische Nachrichten*, 173(1):259–286, 1995.

[19] Motlatsi Molati and Hideki Murakawa. Exact solutions of nonlinear diffusion-convection-reaction equation: A lie symmetry analysis approach. *Communications in Nonlinear Science and Numerical Simulation*, 67(1):253–263, 2019.

[20] Ben Muatjetjeja and Chaudry M. Khalique. Symmetry analysis and conservation laws for a coupled (2+ 1)-dimensional hyperbolic system. *Communications in Nonlinear Science and Numerical Simulation*, 22(1-3):1252–1262, 2015.

[21] Ben Muatjetjeja and Chaudry M. Khalique. Lagrangian formulation of a generalised coupled hyperbolic system. *Journal of King Saud University-Science*, 32(6):2886–2891, 2020.

[22] Peter J. Olver. *Applications of Lie groups to differential equations*. Springer Science & Business Media, 2000.

[23] Levi V. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, New York, 1982.

[24] Vicenţiu D. Rădulescu. *Qualitative analysis of nonlinear elliptic partial differential equations: monotonicity, analytic and variational methods*. Hindawi Publishing Corporation, 2008.

[25] David R. Scott and David J. Stevenson. Magma ascent by porous flow. *Journal of Geophysical Research: Solid Earth*, 91(B9):9283–9296, 1986.

[26] Guido H. Sweers. *Positivity for a strongly coupled elliptic system by Green function estimates*. TU, Faculteit der Technische Wiskunde en Informatica, 1991.