Solving the Barrier Options Model with Linear Time-Dependent Volatility by Symmetry Analysis

Relebohile Veronecca Kometsi (202002742)

Supervisor: Mr Ngaka Nchejane

July 2025

The following dissertation is presented for the degree of Honours in Applied Mathematics in the Department of Mathematics and Computer Science.

Declaration

I, Relebohile Veronecca Kometsi, declare that project entitled, "Solving the barrier options model with linear time-dependent volatility by symmetry analysis " represents my own work which has been done for degree of Honours in Applied Mathematics. I have read the university, faculty, and department regulations, and accept the responsibility for the conduct of the procedures in accordance with university's regulations, and where I have consulted the published work, this has been clearly attributed observing the university policy on plagiarism.

Initial(s): R.V Surname: Kometsi 7th day of July 2025

Abstract

This project aims to solve the barrier options model with linear time-dependent volatility, described as a Partial Differential Equation (PDE) using symmetry analysis. The only PDE to be considered will be the Black-Scholes model. Firstly, we use Lie symmetry to find symmetries of our model from our determining equations. Once the symmetries are found, they will be picked at random to form characteristic equations needed to find invariant solutions. Lastly, the solutions will be interpreted and graphed.

Keywords

- Lie Symmetry Analysis
- Barrier Options Model
- Differential Equations
- Invariant Solutions
- Linear Volatility

Acknowledgements

I would like to express my gratitude to my supervisor Mr. Ngaka Nchejane for his guidance and assistance throughout this dissertation. I also would like to pass my sincere gratitude to Prof. M. Molati and Mr. Mosito Lekhoaoana for the foundation they both instilled in me regarding Lie symmetry analysis. In addition, I pass my sincere thanks to Mrs. 'Mabatho Thakeli for her valuable insights and my classmates for their support.

Contents

List of Figures

Introduction

In financial mathematics, the Black-Scholes model is a fundamental tool for pricing options, providing a closed-form solution for standard vanilla options. However, in the case of barrier options, where the pay-off depends on whether the underlying asset reaches a certain threshold, the problem becomes more complex. This project explores the application of Lie symmetry analysis to solve the Black-Scholes equation for barrier options. Lie symmetry analysis provides a systematic approach to finding exact solutions by identifying symmetries in the governing partial differential equations (PDEs). By applying this method to the Black-Scholes model, our objective is to reduce the complexity of the equation, uncover exact solutions, and enhance the understanding of how barrier options can be priced more effectively in financial markets. Through this approach, we also explore the potential for improving computational efficiency in option pricing algorithms.

Barrier Options

A barrier option is a type of financial derivative whose pay-off depends on whether the price of the underlying asset reaches some specified barrier level during the lifetime of the option. This simply means that the barrier options are path-dependent options. If the barrier is not reached, the option remains active until it expires. If the barrier is reached, the option is immediately terminated (knocked out) or activated (knocked in).

Barrier options are composed of four basic forms:

- down-and-out,
- down-and-in,
- up-and-out,

- up-and-in.

In the 1970s when the Black-Scholes model revolutionised option pricing, that was the beginning of trading Barrier options in the OTC market [1–4]. In the 1990s, when financial markets continued to change, barrier options became more standardised and accessible so that financial institutions and derivatives started to offer them in a wider range [5].

Furthermore, as financial markets became increasingly sophisticated and derivative trading expanded, barrier options gained significant popularity among institutional investors. These instruments offered a cost-effective alternative to standard European options due to their path-dependent structure, which often rendered them less expensive, while still enabling tailored risk exposure. Barrier options allowed users to hedge or speculate based on the belief that an asset would or would not reach a certain price level. This targeted protection was particularly appealing to investment banks, corporate treasuries, and hedge funds engaged in customized risk management and speculative strategies. Their appeal was in the ability to structure payoffs around key levels in the market, making them ideal for scenarios where traditional options were too blunt or expensive a tool [6, 7].

Black-Scholes

The barrier options model is based on Black-Scholes model which estimates the theoretical value of derivatives based on other investment instruments [8–13]. It was first studied by Merton(1973) whereby he priced down-and-out barrier options by solving a transferred stochastic differential equation with boundary equations [2–4]. The model is given by:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0, \tag{1}$$

where:

- $v(s, t)$ = the price of the option as a function of the underlying asset price s and time t,
- s = the current price of the underlying asset,
- t = current time,

- σ = volatility of the underlying asset,
- r = risk-free interest rate,
- $\frac{\partial v}{\partial t}$ = the rate of change of the option price over time (Theta),
- $\frac{\partial v}{\partial s}$ = the sensitivity of the option price to changes in the underlying asset price (Delta),
- $\frac{\partial^2 v}{\partial s^2}$ = the rate of change of the Delta (Gamma).

This model assumes a geometric Brownian motion for the asset price, where the logarithmic returns are normally distributed [14, 15].

However, in real-world markets, volatility often exhibits time dependence, especially under periods of market stress or macroeconomic uncertainty. This has led to various extensions of the classical Black-Scholes model, including models where volatility is a deterministic function of time [14].

In the context of this project, we consider a barrier option pricing model with linear time-dependent volatility, where:

$$\sigma(t) = at + b,$$

and investigate its solution using Lie symmetry analysis.

The Black-Scholes model was devised by Fischer Black , Robert Merton, and Myron Scholes. It was the pioneering mathematical framework for determining the theoretical value of an option contract. The first equation was presented in Black and Scholes' 1973 paper, "The pricing of options and corporate liabilities," published in the journal of political economy. Robert C. Merton assisted in editing the paper and released his own article in The Bell Journal of Economics and Management Science later that year: "Theory of Rational Option Pricing." He expanded the mathematical concepts of the model and introduced the term "Black Scholes Of Options pricing [2, 4, 16].

According to [17], Black-Scholes assumes that stock shares or future contracts will have a lognormal distribution of prices after a change in the market. The equation uses this assumption to derive the call option. It is theoretically possible for option sellers to set rational prices for the options they are selling [18].

The model continues to make the prediction that the prices of heavily traded assets are modeled using a geometric Brownian motion with constant change. Furthermore, the model uses the constant variation of the stock, the value of money in time, the price of the option stake, and the time it takes for an option to expire when applied to a stock option [15].

In addition, the Black Scholes model makes some assumptions such as, no dividends being paid out during the life of an option, random markets, no transaction costs in acquiring an option, and normal distribution of returns of underlying assets [18].

Partial Differential Equations(PDEs)

A partial differential equation is a mathematical relation involving a function of multiple independent variables, which includes the function itself, its partial derivatives, and its independent variables:

$$F\left(x, y, u(x,y), u_x(x,y), u_y(x,y), u_{xx}(x,y), u_{xy}(x,y), u_{yx}(x,y), u_{yy}(x,y)\right) = 0.$$

This is an example of a PDE of order 2.

Building on the foundational Black–Scholes model, more complex financial contracts necessitate modifications to the standard PDE framework. For example, barrier options, which activate or expire when the underlying asset hits a certain price, require PDEs with moving or discontinuous boundary conditions [10, 19]. Similarly, American options, which can be exercised at any time before expiry, lead to free boundary problems and are mathematically represented by variational inequalities rather than standard PDEs [20, 21]. These variations demand specialized numerical techniques, including the penalty method, projected successive over-relaxation (PSOR), and finite difference schemes with early exercise logic [19, 22].

The value of PDEs in finance extends beyond pricing. The solutions of these equations encode key risk sensitivities which measure how option prices respond to changes in inputs like the underlying price, time, or volatility [7, 23]. The derivatives of the solutions are essential for constructing delta-neutral portfolios and for executing dynamic hedging strategies. Furthermore, in the field of portfolio optimization, PDEs appear through the

Hamilton–Jacobi–Bellman (HJB) equation, a fundamental tool in stochastic control theory used to determine optimal investment and consumption policies [24, 25]. Thus, PDEs serve as both descriptive and prescriptive instruments in mathematical finance, bridging theory and practice.

The objectives of this research are to consider the Black-Scholes PDE to incorporate linear-time dependent volatility, determine lie point symmetries using Lie Symmetry analysis and use identified symmetries to explore invariant solutions. This study contributes to both the theoretical and practical advancement of financial mathematics by addressing a more realistic and analytically challenging extension of the Black-Scholes framework. The significance of Lie symmetries lies in their ability to systematically aid in solving differential equations. They do this by yielding first-order differential equations, which can be used to reduce the order or complexity of the equations. In addition, Lie symmetry methods allow for the classification of differential equations based on their symmetry properties. Importantly, the group of invariant transformations admitted by a differential equation can reveal valuable insights about the equation itself and, by extension, the underlying mathematical model [26]. For other applications of Lie symmetry analysis in financial mathematics, one can find them in [27–29].

ii

Lie Symmetry Analysis

Lie symmetry analysis, named after the mathematician Sophus Lie, is a powerful mathematical method used to study and solve partial differential equations analytically. We present a brief summary for finding and using point symmetries to solve differential equations. For further information the reader is referred to [30–32] among others.

Theorems

Definition 1.1.1 A k-th-order ($k \geq 1$) system E of s differential equations (DEs) is defined by

$$E_\sigma\left(x, u, u^{(1)}, \ldots, u^{(k)}\right) = 0, \quad \sigma = 1, \ldots, s, \tag{2}$$

where $u \equiv (u_1, u_2, \ldots, u_q)$ is the dependent vector, $x \equiv (x_1, x_2, \ldots, x_n)$ is the independent vector, and $u^{(1)}, u^{(2)}, \ldots, u^{(k)}$ represent the collection of all first, second, up to k-th order derivatives.

Definition 1.1.2 A symmetry transformation of the system (2) is an invertible transformation of the variables x and u, namely

$$\bar{x}_i = f_i(x, u), \quad \bar{u}_\alpha = \phi_\alpha(x, u), \quad i = 1, \ldots, n; \quad \alpha = 1, \ldots, q, \tag{3}$$

that leaves (2) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E_\sigma(\bar{x}, \bar{u}, \bar{u}^{(1)}, \ldots, \bar{u}^{(k)}) = 0, \quad \sigma = 1, \ldots, s,$$

whenever (2) is satisfied.

Definition 1.1.3 A set G of transformations

$$T_a : \bar{x}_i = f_i(x,u,a), \quad \bar{u}_\alpha = \phi_\alpha(x,u,a), \quad i = 1,\ldots,n; \quad \alpha = 1,\ldots,q, \tag{4}$$

is called a continuous one-parameter (local) Lie-group of transformations in $\mathbb{R}^{n+q}$ provided the group properties of closure, identity, and inverses are satisfied. Here f_i and ϕ_α are differentiable functions, and a is a real parameter which continuously takes values in a neighborhood $D \subseteq \mathbb{R}$ of $a = 0$.

Definition 1.1.4 An infinitesimal generator X of the group transformations G (4) is the differential operator of the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{5}$$

such that

$$\bar{x}_i = x_i + a\xi_i(x,u) + O(a^2) = (1+aX)x_i, \quad \bar{u}_\alpha = u_\alpha + a\eta_\alpha(x,u) + O(a^2) = (1+aX)u_\alpha. \tag{6}$$

Here and throughout this section, the Einstein summation convention is adopted. The one-parameter group elements (6) are known as the infinitesimal transformations obtained from (4) by first-order (Taylor expansion) approximations around parameter $a = 0$. The operator (5) is also called the vector field or the Lie point symmetry generator (or operator).

Definition 1.1.5 The extended infinitesimal generator $X^{[k]}$ of the k-th prolonged (extended) group $G^{[k]}$ on the space $(x, u, \ldots, u^{(k)})$ is called the k-th prolongation of X, viz.,

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u^{(1)})\frac{\partial}{\partial u_i^\alpha} + \cdots + \zeta^\alpha_{(i_1\ldots i_k)}\frac{\partial}{\partial u^\alpha_{(i_1\ldots i_k)}}.$$

The coefficients ζ_s^α are defined recursively by the prolongation formulae:

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u^\alpha_{(j)}D_i(\xi^j),$$

$$\zeta_{ij}^\alpha = D_j(\zeta_i^\alpha) - u^\alpha_{(il)}D_j(\xi^l),$$

$$\vdots$$

$$\zeta^{\alpha}_{i_1\ldots i_k} = D_{i_k}(\zeta^{\alpha}_{i_1\ldots i_{k-1}}) - u^{\alpha}_{(i_1\ldots i_{k-1}l)} D_{i_k}(\xi^l),$$

where the total derivative operator with respect to x^i is given by:

$$D_j = \frac{\partial}{\partial x^j} + u^{\alpha}_{(j)} \frac{\partial}{\partial u^{\alpha}} + u^{\alpha}_{(jk)} \frac{\partial}{\partial u^{\alpha}_{(k)}} + \ldots.$$

Furthermore,

$$u^{\alpha}_{(j)} = D_j(u^{\alpha}), \quad u^{\alpha}_{(jk)} = D_j(u^{\alpha}_{(k)}).$$

Theorem 1.1.1 Let G be a group of infinitesimal transformations admitted by a system E. The group G consists of symmetries of E if and only if the prolonged infinitesimal generator satisfies

$$X^{[k]}\left(E_{\sigma}(x, u, u^{(1)}, \ldots, u^{(k)})\right) = 0, \quad \sigma = 1, \ldots, s, \tag{7}$$

whenever the system E holds. This condition, known as the *invariance criterion*, can be compactly written as

$$X^{[k]}\left(E_{\sigma}(x, u, u^{(1)}, \ldots, u^{(k)})\right)\Big|_{E} = 0, \quad \sigma = 1, \ldots, s, \tag{8}$$

where the notation $\Big|_{E}$ signifies evaluation on the surface defined by E. These equations represented by (8), called the *determining equations*, form the basis of Lie's algorithm.

Generally, the determining equations constitute an overdetermined system of linear homogeneous partial differential equations (PDEs) for the unknown symmetry generator components ξ^i and η^{α}. The solutions of this system form a vector space, meaning that any finite linear combination of symmetries results in another valid symmetry. This property follows from the linear nature of the determining equations. While trivial symmetries, such as translations, are straightforward to identify, more intricate symmetries can be systematically derived using Lie's algorithm.

We will adopt Lie's algorithm as explained in [31].

Lie's algorithm:

1. Write E such that all the terms are on the left-hand side.
2. Write the symmetry generator:

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}.$$

3. Prolong the symmetry generator X to the same order as E, i.e.,

$$X^{[k]} = X + \zeta^i(t, x, u, u^{(1)})\frac{\partial}{\partial u^{(1)}} + \cdots + \zeta^{i_1 \dots i_k}(t, x, u, u^{(1)}, \dots, u^{(k)})\frac{\partial}{\partial u_{(i_1 \dots i_k)}},$$

   where the variables ζ^i are given by (Def 1.1.5).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated at $E = 0$, which gives the symmetry condition:

$$X^{[k]}(E)\Big|_{E=0} = 0.$$

5. Substitute the ζ^i upon expansion of the symmetry condition and replace the derivatives which need elimination.
6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers, resulting in an over-determined system of linear homogeneous PDEs in terms of the infinitesimals ξ^1, ξ^2, and η.
7. Solve the over-determined system for the infinitesimals ξ^1, ξ^2, and η.
8. Construct the one-parameter group.

Invariant solutions definition

Definition 1.2.1. A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point if it remains unchanged by every transformation of a group G, i.e.,

$$(\bar{t}, \bar{x}, \bar{u}) = (f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = (t, x, u), \quad \forall a \in D_0 \subset D.$$

Theorem 1.2.1. A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point of a group G with operator

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u},$$

if and only if

$$\xi^1(t,x,u) = \xi^2(t,x,u) = \eta(t,x,u).$$

Definition 1.2.2. A function $F(t, x, u)$ is called an invariant of a group G if and only if

$$F(\bar{t}, \bar{x}, \bar{u}) = F(f(t,x,u,a), g(t,x,u,a), h(t,x,u,a)) = F(t,x,u),$$

for $t, x, u, a \in D_0 \subset D$.

Theorem 1.2.2. A function $F(t, x, u)$ is an invariant of a group G with generator X if and only if $X(F) = 0$, i.e.,

$$X(F) = \xi^1\frac{\partial F}{\partial t} = \xi^2\frac{\partial F}{\partial x} = \eta\frac{\partial F}{\partial u}.$$

Application of Lie symmetry analysis on our model

We now apply theorems to our model. The goal is to determine the infinitesimals ξ^1, ξ^2, η corresponding to Lie point symmetries of equation (1) given by:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0,$$

i.e, we seek for the generator of Lie point Symmetries of (1) having the form:

$$X = \xi^1(s,t,v)\frac{\partial}{\partial t} + \xi^2(s,t,v)\frac{\partial}{\partial s} + \eta(s,t,v)\frac{\partial}{\partial v}. \tag{9}$$

We act on (9) using:

$$X^{[2]}(v_t + \frac{1}{2}(a^2t^2 + 2atb + b^2)s^2 v_{ss} + rsv_s - rv)|_{(1.8)} = 0, \tag{10}$$

where:

$$X^{[2]} = \xi^1\partial t + \xi^2\partial s + \eta\partial v + \zeta_t\partial vt + \zeta_s\partial vs + \zeta_{tt}\partial v_{tt} + \zeta_t s\partial v_{ts} + \zeta_{ss}\partial v_{ss}, \tag{11}$$

is the second prolongation of (10).

Next, we compute partial derivatives of our model. This are computed as follows:

$$\partial t = (a^2t + ab)s^2 v_{ss}, \tag{12}$$

$$\partial s = sv_{ss}(a^2t^2 + 2atb + b^2) + rv_s, \tag{13}$$

$$\partial V = -r, \tag{14}$$

$$\partial v_t = 1, \tag{15}$$

$$\partial v_s = rs, \tag{16}$$

$$\partial v_{ss} = \frac{1}{2}(a^2t^2 + 2atb + b^2)s^2. \tag{17}$$

Substituting equations (12) to (17) on our second prolongation we get our second prolongations being:

$$X^{[2]} = \xi^1 s^2 v_{ss}(a^2t+ab)+\xi^2 rv_s+\xi^2 sv_{ss}(at+b)^2-\eta r+\zeta_t+\zeta_s rs+\zeta_{ss}\frac{1}{2}s^2(at+b)^2, \tag{18}$$

where:

$$\zeta_t = \eta_t + v_t(\eta_v - \xi^1_t) - v_t^2\xi^1_v - v_s\xi^2_t - v_s v_t\xi^2_v, \tag{19}$$

$$\zeta_s = \eta_s + v_s(\eta_v - \xi^2_s) - v_s^2\xi^2_v - v_s\xi^1_s - v_s v_t\xi^1_v, \tag{20}$$

$$\begin{aligned}\zeta_{ss} = {} & \eta_{ss} + v_s(2\eta_{sv} - \xi^2_{ss}) + v_{ss}(\eta_v - 2\xi^2_s) + v_s^2(\eta_{vv} - 2\xi^2_{sv}) - v_s^3\xi^2_{vv} \\ & - \xi^1_v(v_t v_{ss} + 2v_s v_{st}) - 3v_s v_{ss}\xi^2_v - 2v_{st}\xi^1_s - v_t\xi^1_{ss} - 2v_s v_t\xi^1_{sv} - v_s^2 v_t\xi^1_{vv}\end{aligned} \tag{21}$$

Upon expansion of (18) by substitution of equations 19-21 and replacing v_t by $rv - rsv_s - \frac{1}{2}(at+b)^2s^2v_{ss}$ we get

$$\begin{aligned}
&\xi^1 s^2 v_{ss}(a^2t+ab) + \xi^2 rv_s + \xi^2 s(at+b)^2 v_{ss} - \eta r + \eta_t \\
&+ \left(rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss}\right)(\eta_v - \xi^1_t) \\
&- \left(r^2v^2 + r^2s^2v_s^2 + \tfrac{1}{4}(at+b)^4s^4v_{ss}^2 - 2r^2svv_s - rvs^2(at+b)^2v_{ss} + rs^3(at+b)^2v_sv_{ss}\right)\xi^1_v \\
&- v_s\xi^2_t - \xi^2_v v_s\left(rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss}\right) \\
&+ rs\big[\eta_s + v_s(\eta_v - \xi^2_s) - v_s^2\xi^2_v - v_s\xi^1_s - \xi^1_v v_s\left(rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss}\right)\big] \\
&+ \tfrac{1}{2}s^2(at+b)^2\big[\eta_{ss} + v_s(2\eta_{sv} - \xi^2_{ss}) + v_{ss}(\eta_v - 2\xi^2_s) + v_s^2(\eta_{vv} - 2\xi^2_{sv}) \\
&\qquad - v_s^3\xi^2_{vv} - \xi^1_v\left((rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss})v_{ss} + 2v_sv_{st}\right) \\
&\qquad - 3v_sv_{ss}\xi^2_v - 2v_{st}\xi^1_s - (rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss})\xi^1_{ss} \\
&\qquad - 2v_s(rv - rsV_s - \tfrac{1}{2}(at+b)^2s^2v_{ss})\xi^1_{sv} \\
&\qquad - v_s^2(rv - rsv_s - \tfrac{1}{2}(at+b)^2s^2v_{ss})\xi^1_{vv}\big] = 0.
\end{aligned} \tag{22}$$

Separating (22) by powers of derivative and products of derivatives of v we get:

$$v_s v_{st} : (at+b)^2 s^2 \xi^1_v = 0, \tag{23}$$

$$v_{st} : -(at+b)^2 s^2 \xi^1_s = 0, \tag{24}$$

$$v_s^3 : \frac{1}{2}(at+b)^2 s^2 (-\xi^2_{vv} + \xi^1_{vv} rs) = 0, \tag{25}$$

$$v_s^2 v_{ss} : \frac{1}{4}(at+b)^4 s^4 \xi^1_{vv} = 0, \tag{26}$$

$$v_s v_{ss} : \xi^1_{sv}(at+b)^2 s^2 - 2\xi^2_v (at+b)^2 s^2 = 0, \tag{27}$$

$$v_s^2 : \eta_{vv} s^2 (at+b)^2 - 2\xi^2_{sv}(at+b)^2 s^2 + 2\xi^1_{sv}(at+b)^2 rs^3 - \xi^1_{vv}(at+b)^2 vrs^2 = 0, \tag{28}$$

$$\begin{aligned} v_s : \quad & \xi^2 r + \xi^1_t rs + 2\xi^1_v r^2 vs - \xi^2_t - \xi^2_v rv + \xi^2_s rs + \xi^1_s r^2 s^2 - \xi^1_v r^2 vs \\ & + \tfrac{1}{2}(at+b)^2 \left[2\eta_{vs} - \xi^2_{ss} + \xi^1_{ss} rs - 2\xi^1_{vs} rv\right] = 0, \end{aligned} \tag{29}$$

$$\begin{aligned} v_{ss} : \quad & \xi^1 (a^2 t + ab) s^2 + \xi^2 (at+b)^2 s + \frac{1}{2}\xi^1_t (at+b)^2 s^2 \\ & + \frac{1}{2}\xi^1_v (at+b)^2 s^2 rv + \frac{1}{2}\xi^1_s (at+b)^2 s^3 r + \frac{1}{2}\xi^1_v (at+b)^2 rs^3 \\ & + \frac{1}{2}(at+b)^2 s^2 \left[-2\xi^2_x + \frac{1}{2}\xi^1_{ss}(at+b)^2 s^2 + \xi^1_{sv}(at+b)^2 s^2\right] = 0, \end{aligned} \tag{30}$$

constants:

$$-\eta r + \eta_t + rv(\eta_v - \xi^1_t) - \xi^1_v r^2 v^2 + \eta_s rs - \xi^1_s r^2 vs + (1/2)(at+b)^2 s^2 [\eta_{ss} - \xi^1_{ss} rv] = 0. \tag{31}$$

The coefficients of the infinitesimal generator are:

$$\xi^1(t,s,v) == \frac{C_1}{b} + bC_2 + b^4 C_3, \tag{32}$$

$$\xi^2(t,s,v) == s(C_4 + tC_5), \tag{33}$$

$$\eta(t,s,v) == v(\frac{tC_5}{2} - \frac{rtC_5}{b^2} + C_6 + \frac{C_5 \log[s]}{b^2} + F(s,t)). \tag{34}$$

Our lie symmetries are computed as follows:

$C_1 == 1$ and all the other constants are zeros

$$X_1 = \frac{1}{b}\partial t.$$

$C_2 == 1$

$$X_2 = b\partial t.$$

$C_3 == 1$

$$X_3 = b^4\partial t.$$

$C_4 == 1$

$$X_4 = s\partial s.$$

$C_5 == 1$

$$X_5 = st\partial s + v(\frac{t}{2} - \frac{rt}{b^2} + \frac{\log[s]}{b^2})\partial v.$$

$C_6 == 1$

$$X_6 = v\partial v.$$

Infinite symmetry:

$$X_f = F_1(t,s)\partial v.$$

Invariant solutions

- Invariance under $X_4 + X_6$. The characteristic equations are given by

$$\frac{dt}{0} = \frac{ds}{s} = \frac{dv}{v}, \tag{35}$$

which give similarity variables as $Z_1 = t$ and $Z_2 = \frac{v}{s}$, therefore, the invariant solution $Z_2 = f(Z_1)$ is

$$v(t, s) = sf(t). \tag{36}$$

Substituting (36) into (2) we obtain the following ODE

$$sf'(t) + rsf(t) - rsf(t) = 0, \tag{37}$$

and it solves to $f(t) = c$ where c is an arbitrary constant. Then the invariant solution becomes

$$v(t, s) = sc. \tag{38}$$

- Invariance under $X_2 + X_4$. The characteristic equations are given by

$$\frac{dt}{b} = \frac{ds}{s} = \frac{dv}{0}, \tag{39}$$

which give similarity variables as $Z_1 = se^{\frac{-t}{b}}$ and $Z_2 = v$, therefore, the invariant solution $\mathrm{Z}_2 = f(Z_1)$ is

$$v(t, s) = f(se^{\frac{-t}{b}}). \tag{40}$$

Substituting (40) into (2) we obtain the following ODE

$$\frac{1}{2}(at+b)^2 s^2 e^{\frac{-2t}{b}} f''(se^{\frac{-t}{b}}) + (r - \frac{1}{b})se^{\frac{-t}{b}} f' - rf(se^{\frac{-t}{b}}) = 0. \tag{41}$$

If we let $Z = se^{\frac{-t}{b}}$, equation (41) becomes

$$C_1 Z^2 f''(Z) + C_2 Z f'(Z) + C_3 f(Z) = 0, \tag{42}$$

where:

- $C_1 = \frac{b^2}{2}$, $a = 0$,
- $C_2 = r - \frac{1}{b}$,
- $C_3 = -r$.

Equation (42) is a Euler-Cauchy equation thus we can solve it by assuming a solution $f(Z) = Z^R$. First and second derivative are as follows: $f'(Z) = RZ^{R-1}$ and $f"(Z) = (R-1)(R)Z^{R-2}$.Substituing these into equation (42) we get:

$$C_1 Z^2 R(R-1)Z^{R-2} + C_2 Z R Z^{R-1} + C_3 Z^R. \tag{43}$$

Equation (43) reduces to

$$Z^R[R^2 C_1 + R(C_2 - C_1) + C_3] = 0. \tag{44}$$

Since this has to be true for all Z in the problem domain, we obtain the characteristic equation

$$C_1 R^2 + (C_2 - C_1)R + C_3 = 0. \tag{45}$$

Next we solve the characteristic equation to find R_1 and R_2 using the quadratic equation. The general solution of our equation depends on R_1 and R_2.

- Case 1: Real, distinct roots R_1, R_2,

$$f(se^{\frac{-t}{b}}) = A{se^{\frac{-t}{b}}}^{R_1} + B{se^{\frac{-t}{b}}}^{R_2}. \tag{46}$$

- Case 2: Real, equal roots $R_1 = R_2 = R$,

$$f(se^{\frac{-t}{b}}) = (se^{\frac{-t}{b}})^R(A + B\ln|se^{\frac{-t}{b}}|). \tag{47}$$

- Case 3: roots $R_1 = R_2 = R = \alpha \pm i\beta$,

$$f(se^{\frac{-t}{b}}) = (se^{\frac{-t}{b}})^\alpha(A\cos(\beta\ln|se^{\frac{-t}{b}}|) + Bsin(\beta ln|se^{\frac{-t}{b}}|)), \tag{48}$$

where A and B are arbitrary constants.

- Invariance under $X_1 + X_4$. The characteristic equations are given by

$$\frac{bdt}{1} = \frac{ds}{s} = \frac{dv}{0}, \tag{49}$$

which give similarity variables as $Z_1 = se^{-tb}$ and $Z_2 = v$, therefore the invariant solution $Z_2 = f(Z_1)$ is

$$v(t,s) = f(se^{-tb}). \tag{50}$$

Substituting (50) into (2) we get:

$$\frac{1}{2}(at+b)^2 s^2 e^{-2tb} f'' + (r - \frac{1}{b}) se^{-tb} f' - rf = 0. \tag{51}$$

Similarly we continue with euler-cauchy equation but in this case $Z = se^{-tb}$ and $C_2 = r - b$. However, general solution form is still the same.

- Invariance under $X_2 + X_4 + X_6$. The characteristic equations are given by

$$\frac{dt}{b} = \frac{ds}{s} = \frac{dv}{v} \tag{52}$$

which give similarity variables as $Z_1 = se^{-tb}$ and $Z_2 = \frac{v}{s}$, therefore the invariant solution $Z_2 = f(Z_1)$ is

$$v(t,s) = sf(se^{-tb}). \tag{53}$$

Substituting (53) into (2) we get:

$$\frac{1}{2}(at+b)^2[se^{-2bt} f'' + 2e^{-tb} f'] - bs^2 e^{-tb} f' + rs[f + se^{-tb} f'] - rf. \tag{54}$$

- Invarience under $X_2 + X_5$. The characteristic equations are given by

$$\frac{dt}{b} = \frac{ds}{st} = \frac{dv}{v(\frac{t}{2} - \frac{rt}{b^2} - \frac{log|s|}{b^2})} \tag{55}$$

which give similarity variables $Z_1 = se^{-t^2 2b}$, $\quad Z_2 = \frac{v}{e^{\frac{t^3}{12b} - \frac{rt^3}{3b^3} - \frac{t^2 log|Z_1|}{2b}}}$, therefore the invarient solution $Z_2 = f(Z_1)$ is

$$v(t,s) = e^{\frac{t^3}{12b} - \frac{rt^3}{3b^3} - \frac{t^2 log|se^{-t^2 2b}|}{2b}} f(se^{-t^2 2b}) \tag{56}$$

Substituting (56) into (2) gives a complex ODE which cannot be solved without using mathematical tools being:

$$\boxed{\begin{aligned}&\left(\tfrac{t^2}{4b}-\tfrac{r\,t^2}{b^3}-4b\,t\,\ln|se^{-2bt^2}|+8b^2\,t^3\right)f(z)-4b\,t\,z\,f'(z)\\&+rs\left(-\tfrac{2b\,t^2}{s}f(z)+\tfrac{z}{s}f'(z)\right)+\tfrac{1}{2}\,(at+b)^2s^2\Big[-\tfrac{2b\,t^2}{s}\left(-\tfrac{2b\,t^2}{s}f(z)+\tfrac{z}{s}f'(z)\right)\\&\qquad+\left(\tfrac{2b\,t^2}{s^2}f(z)-\tfrac{2b\,t^2}{s^2}zf'(z)-\tfrac{z}{s^2}f'(z)+\tfrac{1}{s^2}\left(zf'(z)+z^2f''(z)\right)\right)\Big]\\&-r\,f(z)=0,\end{aligned}}$$

where $Z=se^{-2bt^2}$.

Results and Interpretation of solutions

The solutions derived for $v(s,t)$ from the Lie symmetry analysis of the Black-Scholes model can offer insights into the behavior of financial derivatives, including certain types of barrier options. We interpret these forms in the context of key option pricing metrics: Delta, Gamma, and Theta.

Case 1: Distinct Real Roots ($R_1 \neq R_2$)

Solution Form:

$$v(s,t) = As^{R_1}e^{-tbR_1} + Bs^{R_2}e^{\frac{-t}{b}R_2}$$

Interpretation in the Context of Barrier Options:

1. **Behavior with respect to s (Underlying Asset Price) - Delta and Gamma:**

   - This solution often results in a non-monotonic profile for $v(s,t)$ as a function of s, frequently exhibiting a U-shaped or J-shaped curve. This means the option value might decrease for a range of s values before increasing again, or vice-versa.
   - Such non-monotonicity is highly relevant to barrier options and other exotic derivatives.
     - It could represent options like a double no-touch option, which pays a fixed amount if the underlying asset's price stays within a predefined range (between two barriers) until expiration, but becomes worthless if either barrier is touched. The value would be higher within the range and drop towards zero outside.
     - Alternatively, it could model a combination of certain barrier options, where the value is strongly affected by the proximity and interaction with specific price levels.
   - The positive Gamma (upward bowing of the curve) is consistent with many long option positions, indicating that the option's sensitivity to price movements (Delta) increases as the price moves. This is particularly important for barrier options, where Gamma can become very large as the underlying approaches a barrier.

2. **Behavior with respect to t (Time) - Theta:**

   - The time evolution, governed by the exponential terms e^{-tbR_1} and e^{-tbR_2}, leads to a complex interplay of time decay and potential growth components.
   - For barrier options, time decay ($\Theta < 0$) is generally expected as the probability of hitting (or avoiding hitting) a barrier changes over time. However, the exact profile can be intricate. For instance, an option far from its barrier might exhibit typical time decay, but as time passes and the barrier becomes more relevant (e.g., in a "knock-out" option where you want to avoid the barrier), the time value could behave differently.
   - The parameters R_1, R_2, and b would dictate the specific time decay characteristics, which might deviate from simple exponential decay observed in standard options, reflecting the path-dependent nature.

This solution form is particularly intriguing for modeling derivatives with non-standard payoff profiles, making it well-suited for certain barrier option structures or portfolios of such options. Its non-monotonicity with respect to s suggests dependency on price ranges or proximity to specific barrier levels, while its time evolution reflects the complex interplay of time remaining and barrier event probabilities.

Case 2: Repeated Real Roots ($R_1 = R_2 = R$)

Solution Form:

$$v(s,t) = s^R e^{\frac{-t}{b}R}(A + B(\ln|s| + \frac{-t}{b}))$$

1. **Behavior with respect to s (Underlying Asset Price) - Delta and Gamma:**

   - This solution exhibits a monotonic increasing and convex curve with respect to s. The option value increases as s increases, and its rate of increase accelerates.

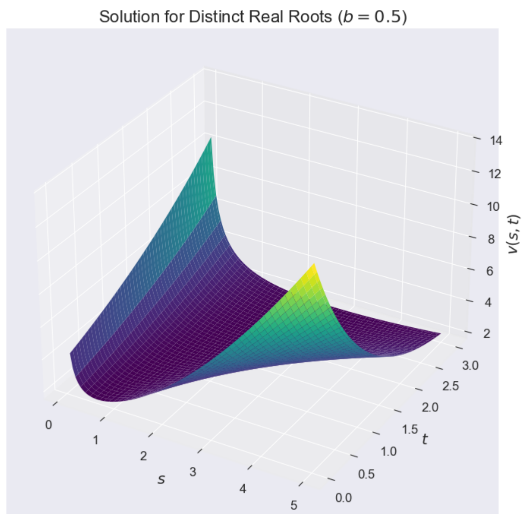
- Financial Analogy: While this strongly resembles a standard European Call Option, it can also represent a barrier option when the underlying price is far away from its barrier. For example, a down-and-out call option would behave almost identically to a standard call when the underlying price s is significantly above the knock-out barrier. In such regions, the probability of the barrier being hit is very low, and thus its impact on the option's value is minimal.
- Delta ($\partial v/\partial s$): Is consistently positive and increasing with s, reflecting the increasing sensitivity of the option's value to favorable movements in the underlying price.
- Gamma ($\partial^2 v/\partial s^2$): Is generally positive, indicating the convexity of the option value. This positive Gamma is crucial for options, as it means their Delta becomes more responsive to price changes.

2. **Behavior with respect to t (Time) - Theta:**

   - The solution demonstrates clear exponential decay in option value as time t increases (assuming $b > 0$ and $R > 0$). This is due to the e^{-tbR} term and the direct $-tb$ within the logarithmic factor.
   - Financial Analogy: This directly corresponds to time decay ($\Theta < 0$), which is a universal characteristic for long positions in both standard and barrier options. As time passes, the period for the underlying asset to move favorably (or for a barrier event to occur) shrinks, leading to a decrease in the option's extrinsic value.

This solution provides a strong representation for a barrier option (such as a down-and-out call) in a region where the underlying price is sufficiently far from the barrier, behaving very similarly to a standard call option. It captures the essential properties of positive Delta, positive Gamma, and typical time decay ($\Theta < 0$).

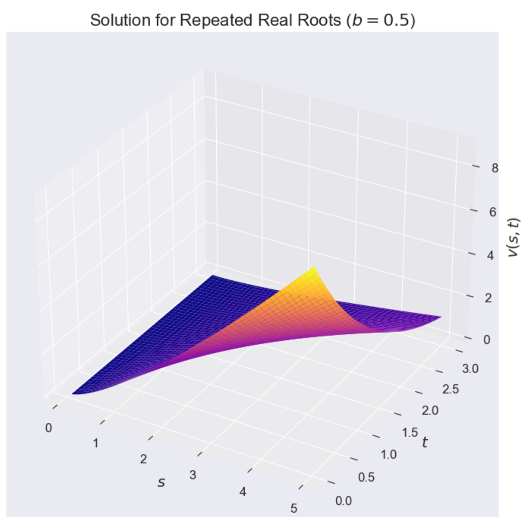
—



(a) Solution Surface for Euler–Cauchy ODE with Distinct Roots.

(b) Solution graph for Euler–Cauchy ODE with Distinct Roots.

(a) Solution graph for Euler–Cauchy ODE with Repeated Roots.



(b) Solution Surface for Euler–Cauchy ODE with Repeated Roots.

Conclusion

In this study, we applied Lie symmetry analysis to a generalized Black-Scholes equation modeling barrier options with linear time-dependent volatility. Through the use of point symmetries, the partial differential equation was systematically reduced to ordinary differential equations. The resulting Euler–Cauchy equation allowed for exact invariant solutions, classified based on the nature of the characteristic roots.

The invariant solutions obtained demonstrated distinct financial behaviors. In the case of distinct roots, the solutions reflected non-monotonic option value profiles, which are particularly suitable for modeling exotic structures such as double-barrier options. The repeated roots case led to smoother, monotonic curves resembling classical call options, especially when far from the barrier.

While the volatility considered was deterministic and linear in time, future work may explore models with stochastic volatility, which more accurately reflect market conditions. Incorporating such dynamics would require the extension of current methods to stochastic PDEs or generalized symmetry frameworks.

Overall, the Lie symmetry method proved to be a powerful analytical tool, offering both mathematical reduction and financial insight into the behavior of barrier options under non-constant volatility.

References

[1] S. Asati, R. Gangele, and B. Kumar, "The black-scholes model: A comprehensive analysis," International Journal on Emerging Technologies, 2024.

[2] F. Black and M. Scholes, "The pricing of options and corporate liabilities," Journal of Political Economy, vol. 81, no. 3, pp. 637–654, 1973.

[3] M. Capiński and E. Kopp, The Black–Scholes Model. Cambridge University Press, 2012.

[4] I. H. Krykun, "The black-scholes exotic barrier option pricing formula," Journal of Mathematics Letters, vol. 1, no. 1, pp. 10–18, 2023.

[5] V. Blanda, "Fx barrier options pricing," p. 1, 2022-2023. http://www.imperial.ac.uk.

[6] E. G. Haug, The Complete Guide to Option Pricing Formulas. New York: McGraw-Hill, 2nd ed., 2007.

[7] J. C. Hull, Options, Futures, and Other Derivatives. Boston: Pearson Education, 9th ed., 2017.

[8] H. Geman and M. Yor, "Pricing and hedging double-barrier options: A probabilistic approach," Mathematical Finance, vol. 6, no. 4, pp. 365–378, 1996.

[9] N. Kunitomo and M. Ikeda, "Pricing options with curved boundaries," Mathematical Finance, vol. 2, no. 4, pp. 275–298, 1992.

[10] E. Reiner and M. Rubinstein, "Barrier options," The Journal of Finance, vol. 46, no. 3, pp. 759–774, 1991.

[11] A. A. J. Pelsser, "Pricing double-barrier options using laplace transforms," Finance and Stochastics, vol. 4, no. 1, pp. 95–104, 2000.

[12] M. Davison and L. Sidogi, "Lie symmetry analysis of barrier option pricing," Applied Mathematics, vol. 4, no. 6, pp. 815–821, 2013.

[13] A. Sagna, "Pricing barrier options by optimal quantization," Methodology and Computing in Applied Probability, vol. 12, no. 3, pp. 465–487, 2010.

[14] Investopedia, "Black–scholes model," 2015. Accessed: 2025-06-17.

[15] Brown University, "Black–scholes model tutorial (mathematica)," 2010. Accessed: 2025-06-17.

[16] R. C. Merton, "Theory of rational option pricing," The Bell Journal of Economics and Management Science, vol. 4, no. 1, pp. 141–183, 1973.

[17] A. Remesh and S. Gaba, "Black-scholes option pricing model: Overview, formula, assumptions," 2024. Accessed: 17 June 2025.

[18] Investopedia, "Black–scholes model," 2023. Accessed: 2025-06-17. Includes explanation of lognormal price assumptions and rational pricing by option sellers.

[19] P. Wilmott, Paul Wilmott on Quantitative Finance. John Wiley & Sons, 2nd ed., 2006.

[20] Y.-K. Kwok, Mathematical Models of Financial Derivatives. Springer, 2nd ed., 2008.

[21] A. Bensoussan and J.-L. Lions, Applications of Variational Inequalities in Stochastic Control. Elsevier, 1984.

[22] S. Ikonen and J. Toivanen, "Operator splitting methods for american option pricing," Applied Mathematics Letters, vol. 17, no. 7, pp. 809–814, 2004.

[23] S. E. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models. Springer, 2004.

[24] H. Pham, Continuous-Time Stochastic Control and Optimization with Financial Applications. Springer, 2009.

[25] T. Björk, Arbitrage Theory in Continuous Time. Oxford University Press, 3rd ed., 2009.

[26] A. Paliathanasis, K. Krishnakumar, K. Tamizhmani, and P. G. Leach, "Lie symmetry analysis of the black-scholes-merton model for european options with stochastic volatility," Mathematics, vol. 4, no. 2, 2016.

[27] Y. Bozhkov and S. Dimas, "Group classification of a generalization of the heath equation," Applied Mathematics and Computation, vol. 243, pp. 121–131, 2014.

[28] T. Motsepa, C. M. Khalique, and M. Molati, "Group classification of a general bond-option pricing equation of mathematical finance," Abstract and Applied Analysis, vol. 2014, p. 709871, 2014.

[29] M. O. Okelola, K. S. Govinder, and J. G. O'Hara, "Solving a partial differential equation associated with the pricing of power options with time-dependent parameters," Mathematical Methods in the Applied Sciences, vol. 38, no. 14, pp. 2901–2910, 2015.

[30] G. Jefferson, Applications of symbolic computing for symmetry analysis of differential equations. PhD thesis, Deakin University, 2014.

[31] M. Khati, Lie group analysis and invariant solutions of a pseudo-parabolic partial differential equation modelling solvent uptake in polymeric solids. PhD thesis, 2020.

[32] L. Mosito, "Lie symmetry analysis and conservation laws of systems of elliptic equations," Master's thesis, National University of Lesotho, 2021.