Solving the barrier options model with linear time-dependent volatility numerically

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Declaration

I, Seeiso Phate, student number 201901836, declare that the research project entitled, *Solving the barrier options model with linear time-dependent volatility numerically* for degree of Bachelor of Science Honours in Applied Mathematics at National University of Lesotho has not been previously submitted by me at this or any other University. Further, I declare that this is my original work and any work done by others has been acknowledged in accordance.

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Abstract

The main purpose of this work is to approximate the solution of the barrier option pricing model whose evolution is described in terms of partial differential equation called Black-Scholes model. In this model, we consider volatility as a linear function of time. This is done primarily using a numerical approximation technique by the name of finite difference method. We consider Crank Nicolson scheme and forward difference scheme in time derivative to discretize the model and represent it as a tridiagonal matrix. Furthermore, we analyse the stability of the discretized model using Von Neumann stability analysis. We finally find the numerical solution to find key insights and the implications.

Keywords

Barrier Options,

Financial Derivatives,

Black-Scholes model,

Option Pricing,

Financial Mathematics,

Partial Differential Equations,

Linear volatility,

Finite Difference Method.

Dedication

To my family and friends.

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Chapter 1

Literature Review

1.1 Barrier Options

Finance is one of the most dynamic and fastest growing segment of the corporate world. Consequently, barrier options are prevalent financial instrument within this field [1–4]. A barrier option is a type of financial derivative contract that is activated or deactivated when the price of the underlying asset (such as stock, commodity, an index, an exchange rate and an interest) reaches or crosses a specific predetermined "barrier" level during the life of the option [5].
An activated option known as knock-in option, is a contract that becomes active only when a certain price is reached. On other hand, the activated option usually called knock-out option, is a contract that starts out as ordinary call or put options, but they become null and void if the spot price ever crosses a certain predetermine level [6, 7].

In addition, an "option" is part of derivative instruments which is a security giving to the owner, right to purchase or sell a specific asset at a set price known as exercise price, subject to particular terms and valid for a limited duration called maturity date [8, 9]. According to Zvan, Robert and Vetzal, Kenneth R and Forsyth, Peter A [2], the demand for barrier options has experienced rapid expansion and there has also been an impressive growth their diversity.
There are barrier options which has two barriers and are known as double barrier options while there are also the ones which are called single barrier options [7,10,11]. Now that we understand barrier options, let us put our focus exclusively in one-factor barrier options

described by partial differential equation called Black-Scholes equation.

1.2 Black Scholes Model

The Black-Scholes model is very popular mathematical model for determining the value of option in financial derivatives. Before 1973, it was a very big problem to predict the value of an option [12]. In the year 1973, Fischer Black and Myron Scholes collaborated to develop the main and original pricing option equation. This equation is used to model pricing options. They further transformed the same pricing option into a new partial differential equation with variable coefficients in the same year [13]. This partial differential equation is called Black-Scholes (coined by Robert Merton) partial differential equation which is mainly used in Financial Engineering [1, 14]. This equation is represented as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

where;
$V(S,t)$ is the value of the option,
$\sigma(t)$ is the volatility,
r is the risk-free interest rate,
S is the price of the underlying asset at a time t, and
t is the time [12].
This mathematical model plays a very vital role in Financial Mathematics because it governs the value of financial derivatives such as options [6].

1.3 Differential Equations

Let us first start by understanding differential equations. Differential equation of a function(or a collection of functions) of a variable (collection of variables) is an algebraic equation that involves both a function and the derivatives with respect to its dependent and independent variables [15]. They are applicable for analyzing and predicting diverse phenomena, such as bank savings growth, the orbit of spaceship, elastic material deformation, the description of radio waves, biological population dynamics, the current or voltage of electric circuit, that is everything that changes continuously [16]. There are often two classes of differential equations being ordinary and partial differential equations of which each arise from only one independent variable and two or more independent variables respectively [17].

The work of M Renardy and RC Rogers [18] tells us that partial differential equations provides fundamental framework for modeling natural phenomena, emerging universally throughout scientific research. In Physics, M Braun [19] stated that Sir Isaac Newton invented differential equations to describe the motion of bodies under gravity. One of the particular area in which partial differential equations play an integral role is in asset pricing theory in general and in pricing of financial derivatives [20]. In this work, we stress more about their contribution in finance.

1.4 Finite Differences

Numerical techniques involve the development and analysis of algorithms for solving equation systems from models of Physics, Biology, Finance phenomena [21–24]. These models considered are composed of set of equations that we mostly do not know how to determine their explicit solutions, and therefore we use numerical methods to obtain an approximate solutions [25, 26].

There are different numerical techniques such as finite element method, finite difference method, boundary element method that are used to approximate the solutions [27–29]. In the work of L.Jing [27], it is stated that the above-mentioned methods can be commonly applied for rock mechanics problems, but in the current study, the focus is on finite difference method. The finite difference method is one of the numerical techniques that solve set of equations such as differential equations [30]. This numerical approach belongs to grid-point methods, wherein a computational domain is covered by space-time grid, with all continuous functions represented discretely at mesh intersection points. The distribution of space-time of the grid points may be arbitrary in principle but it significantly affects the accuracy of the approximation [31]. The finite difference approximations of derivatives are one of the simplest and and the oldest methods to use when solving differential equations.

The main idea is to discretize the derivative in space and time. This happens to a differentiable functions. Both spatial and time domains are broken down into finite number of intervals called grid points, and the values of the solution at these intervals are approximated. In finite difference schemes, the original differential equation is transformed into a set of algebraic equations that can be represented in matrix form and solved using linear algebra techniques [32–34].

1.4.1 Mesh Size

Mesh size / step size (often denoted by h) is very crucial in numerical analysis. A mesh is a discretized representation of a physical domain, broken into small elements. The mesh size refers to the characteristic dimension of these elements. Mesh size arise from grid generation. Chawner [35] tells us that numerical grid generation is now a common tool that is fairly used in the numerical solution of partial differential equations for discretization. According to Timothy J. Baker [36], the criterion of an approximate computational mesh was historically regarded to be a tedious exercise involved in solving partial differential equations by either a finite difference or finite element method. However, mesh generation has since matured into an independent research discipline in its own right drawing on ideas from other fields, in particular Mathematics and Computer Science while cultivating its own unique methodological framework. When the mesh size is small, it resolves the fine details of the geometry, but larger sizes reduces the total number of nodes while improving efficiency and accuracy in discretizations and well conditioned linear systems but introducing complexity in computation [37].

1.4.2 Stability Analysis

Stability is a well-defined and foundational concept across multiple disciplines, including Mathematics, Engineering, Physics, and Dynamical Systems. Its theoretical framework is both compact relying on core principles that apply broadly and consistent, with precise definitions that ensure logical coherence. The study of stability is supported by a rigorous Mathematical Foundation, drawing from: Dynamical Systems, Control Theory, Numerical Analysis and many more [38].

Joseph F. Grcar [39] states that in 1947 work of Von Neumann and Goldstine framed stability as a critical lens for analyzing computational errors, separating the inherent stability of a continuous mathematical problem from the stability of its discrete numerical approximation. By invoking the Courant-Friedrichs-Lewy (CFL) condition, they also demonstrated that even stable continuous problems could produce unstable numerical results if discretization violates stability constraints, highlighting a foundational challenge in scientific computing. Thus analyzing stability will be one of the crucial aspects in the context.

This project focuses on solving barrier options model with time-dependent volatility using numerical technique. The model we are going to focus on is the partial differential equation called **Black-Scholes model** represented by (1.1).

1.5 Aim and Objective

The fundamental aim of this work is to obtain the numerical solution (approximation) of the model. The aim will be accomplished through the following objectives.

1. Discretization of Black-Scholes with linear time-dependent volatility model using finite difference method.
2. Stability analysis.
3. Find the results, key insights from the results and the implications.

Chapter 2

Methods

2.1 Finite Difference

In finite difference, we approximate the derivatives of partial differential equations using discrete differences. Numerical differentiation is the main tool for finite difference method. Given a sufficiently smooth function $f(x)$, we have to find the approximation to $f'(x)$ or $f''(x)$ in some given point x, just by using evaluation of the function itself. The derivative of $f(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Given a sufficiently small value of h, the right hand side can be written as

$$\frac{f(x+h) - f(x)}{h},$$

which is used as an approximation to $f'(x)$.

Under the finite difference method, we subdivide the interval into sub-intervals where the length of each sub-interval is called **the step/mesh-length/size**. Given the interval $[a, b]$, the step-size is denoted by ΔX or h so that

$$\Delta X = h = \frac{b-a}{n}.$$

The points $a = x_0$, $x_1 = x_0 + h$, $x_2 = x_0 + 2h$, ..., $x_n = x_0 + nh$ are called **nodal points**. The small collection of the mostly used approximations to function derivatives are **forward difference scheme**, **backward difference scheme**, **central difference scheme** and **Crank Nicolson scheme**.

2.1.1 Forward Difference Scheme:

$$f'(x) = \frac{dy}{dx} = y' \approx \frac{y(x_{i+1}) - y(x_i)}{h} \approx \frac{y_{i+1} - y_i}{h}$$

and

$$\frac{\partial f}{\partial t} \approx \frac{f(x_i, t_{n+1}) - f(x_i, t_n)}{h_1},$$

where $f = f(x, t)$ and $h_1 = \Delta t$.

2.1.2 Backward Difference Scheme:

$$f'(x) = \frac{dy}{dx} = y' \approx \frac{y(x_i) - y(x_{i-1})}{h} \approx \frac{y_i - y_{i-1}}{h}$$

and

$$\frac{\partial f}{\partial t} \approx \frac{f(x_i, t_n) - f(x_{i-1}, t_n)}{h_2},$$

where $f = f(x, t)$ and $h_2 = \Delta x$.

2.1.3 Central Difference Scheme:

First Derivative:

$$f'(x) = \frac{dy}{dx} = y' \approx \frac{y(x_{i+1}) - y(x_{x-i})}{2h} \approx \frac{y_{i+1} - y_{i-1}}{2h}$$

and

$$\frac{\partial f}{\partial x} \approx \frac{f(x_{i+1}, t_n) - f(x_{i-1}, t_n)}{2h},$$

where $f = f(x, t)$ and $h = \Delta x$.

Second Derivative:

$$f'(x) = \frac{dy}{dx} = y' \approx \frac{y(x_{i+1}) - 2y(x_i) + y(x_{x-i})}{h^2} \approx \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

and

$$\frac{\partial f}{\partial x} \approx \frac{f(x_{i+1}, t_n) - 2f(x_i, t_n) + f(x_{i-1}, t_n)}{h^2},$$

where $f = f(x, t)$ and $h = \Delta x$.

2.1.4 Crank-Nicolson Scheme

This scheme averages the explicit and the implicit schemes for stability/accuracy.

2.2 Stability

Von Neumann stability analysis is also known as Fourier stability analysis. It was invented by Hungarian Mathematician and father of electronic computers by the name of John Von Neumann [40].

2.2.1 Problem

Von Neumann stability addresses the concern of growth of round off errors and/or initially small fractuations in initial data which might cause a large deviation of final answers from exact solution.

This method of stability analysis is based on decomposition of numerical errors of numerical approximations into Fourier series.

Fourier series decomposes any periodic functions or periodic signal into the sum of a (possibly infinite) set of simple oscillating functions called sines and cosines (or, equivalently, complex exponentials).

2.3 Application of Finite Difference on Current Study

Recall that our study is based on **Black-Scholes model** represented as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \, \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$

where time dependent volatility is given as

$$\sigma(t) = at + b. \tag{2.2}$$

Now this means that

$$\sigma^2 = a^2t^2 + 2abt + b^2. \tag{2.3}$$

Now substituting equation (2.3) into (2.1) yields

$$\frac{\partial V}{\partial t} + \frac{1}{2}(a^2t^2 + 2abt + b^2)S^2 \, \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{2.4}$$

2.3.1 Discretazation

We need to discretize equation (2.4) using **finite difference method.**

Time derivative (Forward difference method)

$$\frac{\partial V}{\partial t} \approx \frac{V_i^{n+1} - V_i^n}{\Delta t}. \tag{2.5}$$

First spatial derivative (Central difference method)

$$\frac{\partial V}{\partial S} \approx \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta S}. \tag{2.6}$$

Second spatial derivative (Central difference Method)

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{i+1}^n - 2V_i^n + V_{i-1}^n}{(\Delta S)^2}. \tag{2.7}$$

In our study, we focus on using **Crank-Nicolson Method** on (2.6) and (2.7) respectively.

First spatial derivative (Crank-Nicolson method)

$$\frac{\partial V}{\partial S} \approx \frac{\left[V_{i+1}^{n+1} - V_{i-1}^{n+1}\right] + \left[V_{i+1}^n - V_{i-1}^n\right]}{4\Delta S}. \tag{2.8}$$

Second spatial derivative (Crank-Nicolson method)

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{\left[V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1}\right] + \left[V_{i+1}^n - 2V_i^n + V_{i-1}^n\right]}{2(\Delta S)^2}. \tag{2.9}$$

Applying **forward difference method** in (2.5) and **Crank-Nicolson method** on (2.4) to discretize we obtain

$$\begin{aligned} &\frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{1}{2}(a^2t_n^2 + 2abt_n + b^2)S_i^2\left[\frac{V_{i+1}^{n+1} - 2V_i^{n+1} + V_{i-1}^{n+1} + V_{i+1}^n - 2V_i^n + V_{i-1}^n}{2(\Delta S)^2}\right] \\ &+ rS_i\left[\frac{V_{i+1}^{n+1} - V_{i-1}^{n+1} + V_{i+1}^n - V_{i-1}^n}{4\Delta S}\right] - r\left[\frac{V_i^{n+1} + V_i^n}{2}\right] = 0, \end{aligned} \tag{2.10}$$

which simplifies to

$$\begin{aligned} &\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_{i-1}^{n+1} - \frac{\Delta t}{4(\Delta S)}rS_iV_{i-1}^{n+1} \\ &+ V_i^{n+1} - \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_i^{n+1} - \frac{\Delta t}{2}rV_i^{n+1} \\ &+ \frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_{i+1}^{n+1} + \frac{\Delta t}{4(\Delta S)}rS_iV_{i+1}^{n+1} \\ &= -\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_{i-1}^n + \frac{\Delta t}{4(\Delta S)}rS_iV_{i-1}^n \\ &+ V_i^n + \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_i^n + \frac{\Delta t}{2}rV_i^n \\ &- \frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2V_{i+1}^n - \frac{\Delta t}{4(\Delta S)}rS_iV_{i+1}^n. \end{aligned} \tag{2.11}$$

Upon factorizing we obtain

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2-\frac{\Delta t}{4(\Delta S)}rS_i\right]V_{i-1}^{n+1}\\
&+\left[1-\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2-\frac{\Delta t}{2}r\right]V_i^{n+1}\\
&+\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2+\frac{\Delta t}{4(\Delta S)}rS_i\right]V_{i+1}^{n+1}\\
&=\left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2+\frac{\Delta t}{4(\Delta S)}rS_i\right]V_{i-1}^{n}\\
&+\left[1+\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2+\frac{\Delta t}{2}r\right]V_i^{n}\\
&-\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2+\frac{\Delta t}{4(\Delta S)}rS_i\right]V_{i+1}^{n}.
\end{aligned}\tag{2.12}$$

Now let us consider (2.12) for different values of $i = 1, 2, 3, \ldots, N-1, N$.

For $i = 1$

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2-\frac{\Delta t}{4(\Delta S)}rS_1\right]V_{0}^{n+1}\\
&+\left[1-\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2-\frac{\Delta t}{2}r\right]V_1^{n+1}\\
&+\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2+\frac{\Delta t}{4(\Delta S)}rS_1\right]V_{2}^{n+1}\\
&=\left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2+\frac{\Delta t}{4(\Delta S)}rS_1\right]V_{0}^{n}\\
&+\left[1+\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2+\frac{\Delta t}{2}r\right]V_1^{n}\\
&-\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_1^2+\frac{\Delta t}{4(\Delta S)}rS_1\right]V_{2}^{n},
\end{aligned}\tag{2.13}$$

For $i = 2$

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2-\frac{\Delta t}{4(\Delta S)}rS_2\right]V_{1}^{n+1}\\
&+\left[1-\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2-\frac{\Delta t}{2}r\right]V_2^{n+1}\\
&+\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2+\frac{\Delta t}{4(\Delta S)}rS_2\right]V_{3}^{n+1}\\
&=\left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2+\frac{\Delta t}{4(\Delta S)}rS_2\right]V_{1}^{n}\\
&+\left[1+\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2+\frac{\Delta t}{2}r\right]V_2^{n}\\
&-\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_2^2+\frac{\Delta t}{4(\Delta S)}rS_2\right]V_{3}^{n},
\end{aligned}\tag{2.14}$$

For $i = 3$

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 - \frac{\Delta t}{4(\Delta S)}rS_3\right] V_2^{n+1} \\
&+ \left[1 - \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 - \frac{\Delta t}{2}r\right] V_3^{n+1} \\
&+ \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 + \frac{\Delta t}{4(\Delta S)}rS_3\right] V_4^{n+1} \\
&= \left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 + \frac{\Delta t}{4(\Delta S)}rS_3\right] V_2^{n} \\
&+ \left[1 + \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 + \frac{\Delta t}{2}r\right] V_3^{n} \\
&- \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_3^2 + \frac{\Delta t}{4(\Delta S)}rS_3\right] V_4^{n},
\end{aligned} \tag{2.15}$$

$$\vdots$$

For $i = N - 1$

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 - \frac{\Delta t}{4(\Delta S)}rS_{N-1}\right] V_{N-2}^{n+1} \\
&+ \left[1 - \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 - \frac{\Delta t}{2}r\right] V_{N-1}^{n+1} \\
&+ \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 + \frac{\Delta t}{4(\Delta S)}rS_{N-1}\right] V_N^{n+1} \\
&= \left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 + \frac{\Delta t}{4(\Delta S)}rS_{N-1}\right] V_{N-2}^{n} \\
&+ \left[1 + \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 + \frac{\Delta t}{2}r\right] V_{N-1}^{n} \\
&- \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_{N-1}^2 + \frac{\Delta t}{4(\Delta S)}rS_{N-1}\right] V_N^{n},
\end{aligned} \tag{2.16}$$

and For $i = N$

$$\begin{aligned}
&\left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 - \frac{\Delta t}{4(\Delta S)}rS_N\right] V_{N-1}^{n+1} \\
&+ \left[1 - \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 - \frac{\Delta t}{2}r\right] V_N^{n+1} \\
&+ \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 + \frac{\Delta t}{4(\Delta S)}rS_N\right] V_{N+1}^{n+1} \\
&= \left[-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 + \frac{\Delta t}{4(\Delta S)}rS_N\right] V_{N-1}^{n} \\
&+ \left[1 + \frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 + \frac{\Delta t}{2}r\right] V_N^{n} \\
&- \left[\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_N^2 + \frac{\Delta t}{4(\Delta S)}rS_N\right] V_{N+1}^{n}.
\end{aligned} \tag{2.17}$$

2.3.2 Matrix Representation

We can represent the system of equations above as tridiagonal matrix form as

$$\begin{bmatrix} B_1 & C_1 & 0 & \cdots & 0 & 0 \\ A_2 & B_2 & C_2 & \cdots & 0 & 0 \\ 0 & A_3 & B_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{N-1} & C_{N-1} \\ 0 & 0 & 0 & \cdots & A_N & B_N \end{bmatrix} \begin{bmatrix} V_1^{n+1} \\ V_2^{n+1} \\ V_3^{n+1} \\ \vdots \\ V_{N-1}^{n+1} \\ V_N^{n+1} \end{bmatrix} = \begin{bmatrix} E_1 & F_1 & 0 & \cdots & 0 & 0 \\ D_2 & E_2 & F_2 & \cdots & 0 & 0 \\ 0 & D_3 & E_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & E_{N-1} & F_{N-1} \\ 0 & 0 & 0 & \cdots & D_N & E_N \end{bmatrix} \begin{bmatrix} V_1^{n} \\ V_2^{n} \\ V_3^{n} \\ \vdots \\ V_{N-1}^{n} \\ V_N^{n} \end{bmatrix}$$

$$+ \begin{bmatrix} -A_1(V_0^n + V_0^{n+1}) \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -C_N(V_{N+1}^n + V_{N+1}^{n+1}) \end{bmatrix}, \tag{2.18}$$

where the elements are defined as:

$$\begin{aligned} A_i &= \frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 - \frac{\Delta t}{4\Delta S}rS_i, \quad i = 2, 3, \ldots, N, \\ B_i &= 1 - \frac{\Delta t}{(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 - \frac{\Delta t}{2}r, \quad i = 1, 2, \ldots, N, \\ C_i &= \frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 + \frac{\Delta t}{4\Delta S}rS_i, \quad i = 1, 2, \ldots, N-1, \\ D_i &= -\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 + \frac{\Delta t}{4\Delta S}rS_i, \quad i = 2, 3, \ldots, N, \\ E_i &= 1 + \frac{\Delta t}{(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 + \frac{\Delta t}{2}r, \quad i = 1, 2, \ldots, N, \\ F_i &= -\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2 + 2abt_n + b^2)S_i^2 - \frac{\Delta t}{4\Delta S}rS_i, \quad i = 1, 2, \ldots, N-1, \end{aligned}$$

and $V_0^n, V_0^{n+1}, V_{N+1}^n, V_{N+1}^{n+1}$ are boundary values at spatial indices 0 and $N+1$ at times n and $n+1$ respectively.

We can see that equation (2.12) is discretized and is of a type

$$\mathbf{M^{n+1}V^{n+1}} = \mathbf{N^nV^n} + \boldsymbol{b}. \tag{2.19}$$

The tridiagonal matrices in (2.18) can be very large. Very large matrices occupy huge accounts of memory, and processing them can take much computer time. These matrices can be solved by making use of iterative methods or the Thomas algorithm [6].

2.3.3 Stability

Von Neumann Stability

For this work, we have chosen complex exponentials to represent errors as they are much better (easier) to work with compared to trigonometric functions.

Recall that we have discretized Black-Scholes equation from (2.11) is

$$\begin{aligned}
&\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_{i-1}^{n+1}-\frac{\Delta t}{4(\Delta S)}rS_iV_{i-1}^{n+1}\\
&+V_i^{n+1}-\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_i^{n+1}-\frac{\Delta t}{2}rV_i^{n+1}\\
&+\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_{i+1}^{n+1}+\frac{\Delta t}{4(\Delta S)}rS_iV_{i+1}^{n+1}\\
&=-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_{i-1}^{n}+\frac{\Delta t}{4(\Delta S)}rS_iV_{i-1}^{n}\\
&+V_i^{n}+\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_i^{n}+\frac{\Delta t}{2}rV_i^{n}\\
&-\frac{\Delta t}{4(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2V_{i+1}^{n}-\frac{\Delta t}{4(\Delta S)}rS_iV_{i+1}^{n}.
\end{aligned} \tag{2.20}$$

We now let

$$\alpha=\frac{\Delta t}{2(\Delta S)^2}(a^2t_n^2+2abt_n+b^2)S_i^2, \tag{2.21}$$

$$\beta=\frac{\Delta t}{2(\Delta S)}rS_i, \tag{2.22}$$

and

$$\gamma=\frac{\Delta t}{2}r. \tag{2.23}$$

This simply means that we can rewrite equation (2.20) as

$$\begin{aligned}
&\frac{\alpha}{2}V_{i-1}^{n+1}-\frac{\beta}{2}V_{i-1}^{n+1}+V_i^{n+1}-\alpha V_i^{n+1}-\gamma V_i^{n+1}+\frac{\alpha}{2}V_{i+1}^{n+1}+\frac{\beta}{2}V_{i+1}^{n+1}\\
&=-\frac{\alpha}{2}V_{i-1}^{n}+\frac{\beta}{2}V_{i-1}^{n}+V_i^{n}+\alpha V_i^{n}+\gamma V_i^{n}-\frac{\alpha}{2}V_{i+1}^{n}-\frac{\beta}{2}V_{i+1}^{n}.
\end{aligned} \tag{2.24}$$

We now factorize (2.24) and obtain

$$\begin{aligned}
&\frac{1}{2}(\alpha-\beta)V_{i-1}^{n+1}+(1-\alpha-\gamma)\,V_i^{n+1}+\frac{1}{2}(\alpha+\beta)V_{i+1}^{n+1}\\
&=-\frac{1}{2}(\alpha-\beta)V_{i-1}^{n}+(1+\alpha+\gamma)\,V_i^{n}-\frac{1}{2}(\alpha+\beta)V_{i+1}^{n}.
\end{aligned} \tag{2.25}$$

We now switch our focus to detailed **stability analysis** of equation (2.25). We basically base ourselves on **Von Neumann stability** analysis which is also known as **Fourier stability** analysis. This method is based on the assumption that the numerical scheme admits a solution of the form

$$\epsilon_i^n = g^n e^{jki\Delta S}, \tag{2.26}$$

where ϵ_i^n is an error at spatial point i and time step n,
g is an **amplification factor** (determines error growth),
k is the **wave number** (frequency of the error),
j is the imaginary unit ($\sqrt{-1}$) and,
ΔS is the spatial step size.
We now have to express V_i^n and V_i^{n+1} as Fourier modes. We assume that the numerical solution has the same form as the error, that is

$$\begin{aligned} V_i^n &= g^n e^{jki\Delta S} \quad , \quad V_i^{n+1} = g^{n+1} e^{jki\Delta S} \\ V_{i-1}^n &= g^n e^{jk(i-1)\Delta S} = g^n e^{jki\Delta S} e^{-jk\Delta S} \\ V_{i+1}^n &= g^n e^{jk(i+1)\Delta S} = g^n e^{jki\Delta S} e^{jk\Delta S}. \end{aligned} \tag{2.27}$$

Substituting set of equations in (2.27) back into equation (2.25) yields

$$\begin{aligned} &\frac{1}{2}(\alpha-\beta) g^{n+1} e^{jki\Delta S} e^{-jk\Delta S} + (1-\alpha-\gamma)\, g^{n+1} e^{jki\Delta S} + \frac{1}{2}(\alpha+\beta) g^{n+1} e^{jki\Delta S} e^{jk\Delta S} \\ &= -\frac{1}{2}(\alpha-\beta) g^{n} e^{jki\Delta S} e^{-jk\Delta S} + (1+\alpha+\gamma)\, g^{n} e^{jki\Delta S} - \frac{1}{2}(\alpha+\beta) g^{n} e^{jki\Delta S} e^{jk\Delta S}. \end{aligned} \tag{2.28}$$

Dividing equation (2.28) by $e^{jki\Delta S}$ yields

$$\begin{aligned} &\frac{1}{2}(\alpha-\beta) g^{n+1} e^{-jk\Delta S} + (1-\alpha-\gamma)\, g^{n+1} + \frac{1}{2}(\alpha+\beta) g^{n+1} e^{jk\Delta S} \\ &= -\frac{1}{2}(\alpha-\beta) g^{n} e^{-jk\Delta S} + (1+\alpha+\gamma)\, g^{n} - \frac{1}{2}(\alpha+\beta) g^{n} e^{jk\Delta S}. \end{aligned} \tag{2.29}$$

Collecting like terms and factorizing equation (2.29) above yields

$$\begin{aligned} &\frac{1}{2}\alpha g^{n+1}(e^{jk\Delta S} + e^{-jk\Delta S}) + \frac{1}{2}\beta g^{n+1}(e^{jk\Delta S} - e^{-jk\Delta S}) + (1-\alpha-\gamma)\, g^{n+1} \\ &= -\frac{1}{2}\alpha g^{n}(e^{jk\Delta S} + e^{-jk\Delta S}) - \frac{1}{2}\beta g^{n}(e^{jk\Delta S} - e^{-jk\Delta S}) + (1+\alpha+\gamma)\, g^{n}. \end{aligned} \tag{2.30}$$

Recall the Euler's identities,

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta \text{ and } e^{j\theta} - e^{-j\theta} = 2j\sin\theta. \tag{2.31}$$

Using the same analogy, we have that

$$e^{jk\Delta S} + e^{-jk\Delta S} = 2\cos(k\Delta S) \text{ and } e^{jk\Delta S} - e^{-jk\Delta S} = 2j\sin(k\Delta S). \tag{2.32}$$

Substituting equations in (2.32) into (2.30) yields

$$\begin{aligned}&\frac{1}{2}\alpha g^{n+1}(2\cos(k\Delta S)) + \frac{1}{2}\beta g^{n+1}(2j\sin(k\Delta S)) + (1-\alpha-\gamma)\, g^{n+1}\\ &= -\frac{1}{2}\alpha g^{n}(2\cos(k\Delta S)) - \frac{1}{2}\beta g^{n}(2j\sin(k\Delta S)) + (1+\alpha+\gamma)\, g^{n}.\end{aligned} \tag{2.33}$$

Factorizing and simplifying equation (2.33) yields

$$\begin{aligned}&[\alpha(\cos(k\Delta S)) + \beta(j\sin(k\Delta S)) + (1-\alpha-\gamma)]\, g^{n+1}\\ &= [-\alpha(\cos(k\Delta S)) - \beta(j\sin(k\Delta S)) + (1+\alpha+\gamma)]\, g^{n}.\end{aligned} \tag{2.34}$$

Now let

$$A = \cos(k\Delta S) \quad \text{and} \quad B = \sin(k\Delta S), \tag{2.35}$$

this simply means that we can rewrite equation (2.34) as

$$\begin{aligned}&[\alpha A + \beta j B + (1-\alpha-\gamma)]\, g^{n+1}\\ &= [-\alpha A - \beta j B + (1+\alpha+\gamma)]\, g^{n}\end{aligned} \tag{2.36}$$

using equations in (2.35).

We further simplify equation (2.36) to have

$$\frac{g^{n+1}}{g^{n}} = g = \frac{[1-\alpha A - \beta j B + \alpha + \gamma]}{[1+\alpha A + \beta j B - \alpha - \gamma]}. \tag{2.37}$$

The strict Von Neumann stability condition is given by

$$|g| \leq 1.$$

This means that equation (2.37) can be written as

$$\begin{aligned}|g| &= \left|\frac{1-\alpha A - \beta j B + \alpha + \gamma}{1+\alpha A + \beta j B - \alpha - \gamma}\right| \leq 1\\ &\implies |1-\alpha A - \beta j B + \alpha + \gamma| \leq |1+\alpha A + \beta j B - \alpha - \gamma|\\ &\implies (1 - A\alpha + \alpha + \gamma)^2 \leq (1 + \alpha A - \alpha - \gamma)^2,\end{aligned} \tag{2.38}$$

which simplifies to

$$(1 - A\alpha)(\alpha + \gamma) \leq 0. \tag{2.39}$$

From an inequality (2.39), we have stability if and only if an inequality is true and instability otherwise.

We see that the stability condition depends on the values of α, A and γ. This simply means that we have to consider all the cases involved to make (2.39) true.

Case 1: If $(\alpha + \gamma) > 0$ then

$$1 - A\alpha \leq 0 \implies A\alpha \geq 1.$$

Case 2: If $(\alpha + \gamma) < 0$ then

$$1 - A\alpha \geq 0 \implies A\alpha \leq 1.$$

Case 3: Special case, If $(\alpha + \gamma) = 0$ (trivial) then the condition is automatically satisfied.

Interpretation

Case 1:

The product of time-stepping parameter (α) and the system parameter (A) must sufficiently large to prevent instability.

Case 2:

The product of time-stepping parameter (α) and the system parameter (A) must sufficiently small to avoid runaway growth.

Case 3:

This case is trivial since the condition is automatically satisfied.

Otherwise we have instability.

Conclusion

We conclude that we have **conditional stability** because stability depends on conditions of inequality (2.39) being satisfied. This is because if the conditions are violated for some parameter choices then we have instability.

Chapter 3

Results And Discussion

3.1 Introduction

In this chapter, we provide a graphical solution of option prices under Black-Scholes model that was considered in the previous chapter. We make the assumptions for parameters for plotting and then give a 2-dimensional graph of matrix (2.18). We finally make use the resulting graph to discuss the behavior of option value with underlying stock price for given time.

3.2 Graphical Solution

The graph below depicts the evolution of option value against stock price under time-varying volatility conditions.

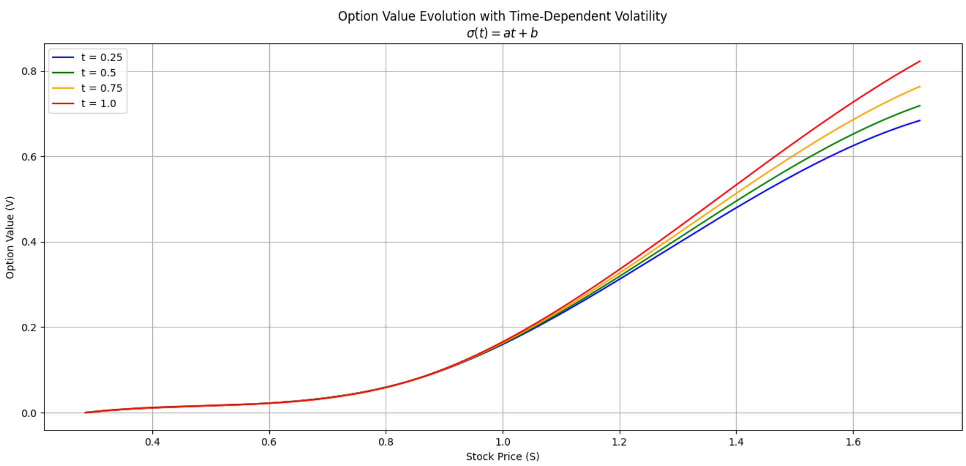

Figure 3.1: Stock price against option value in different times

3.3 Graph Discussion

In Figure 3.1 above, we generally observe that the value of an option increases with stock price for each time. Since the value of an option dependents on stock price and time, we observe that for earliest time, the curve is steeper, implying lower initial volatility which leads to sharper price sensitivity, and at the latest time, the curve is flatter, suggesting higher volatility which smooths out price impacts. The development of trend of Figure 3.1 mimic the behavior of European option and Black-Scholes model numerical solution graph in [41] and [12] respectively.

Chapter 4

Conclusion

This piece of work is another contribution in the area of Financial Mathematics. In this research project, we successfully discretized Black-Scholes model with linear time dependent volatility using finite difference method. We further represented the discretized model in matrix form, analyzed its stability through Von-Neumann analysis and finally found the graphical solution using python within a certain range by making assumptions of parameters.

Moreover, we observed the consistency with call option, suggesting the sellers to sell an option when the stock is at its peak while buyers face reduced upside potential. In addition, it is advisable to sell an option at latest time since the option value will be at its highest. Therefore, we conclude that the objective is accomplished. This accomplishment provides a robust foundation for understanding finite difference method and the underlying model. Finite difference method is valuable because it provides computational efficiency and a clear framework for analyzing stability which is crucial in Financial Mathematics.

For future work, we shall consider quadratic or exponential time dependent volatility and compare the results for different parameters such as interest rate.

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