Symmetry Analysis of the Parabolic System for European Option Pricing with Liquidity Shocks

by

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DECLARATION

I, Bonang Moreboli, declare that this dissertation entitled "Symmetry Analysis of the Parabolic System for European Option Pricing with Liquidity Shocks" represents my work, which has been done for the degree of Master of Science in Mathematics (Applied). I have read the university, faculty, and department regulations and accept the responsibility for the conduct of the procedures in accordance with university regulations, and where I have consulted the published work, this has been clearly attributed to observing the university policy on plagiarism.

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ABSTRACT

We address the knowledge gap between financial theory and practice by developing a framework to analyse European option pricing under liquidity shocks, which reduce market completeness and challenge traditional models like Black–Scholes.To tackle this, we undertake a Lie symmetry analysis of a coupled system designed to model European options subject to liquidity shocks. The system consists of a degenerate parabolic equation and another is a first-order nonlinear ordinary differential equation (ODE) in time with no spatial derivatives. Our analysis has uncovered the parabolic system's Lie symmetry group and infinitesimal generators. We have then proceeded to investigate the system dynamics by constructing commutative tables that illustrate the relationships between the vector fields under study and the one-dimensional optimal system of symmetry subalgebras associated with the initial equation. Employing similarity reductions, we have applied the Lie symmetry methodology to decompose the system into several nonlinear ordinary differential equations (ODEs) corresponding to each symmetry subalgebra. Finally, we present obtained invariant solutions through simulations as 2D and 3D graphs.

KEYWORDS

European Options

Lie Symmetry Analysis

Optimal System

Coupled System

Liquidity Shocks

Partial Differential Equations

Mathematics of Finance

Dedicated to,

My beloved family.

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Table of Contents

List of Figures

List of Tables

1

Introduction

The pricing of European options remains a key area of theoretical and practical interest. Their fundamental role in financial markets and the rich mathematical frameworks for their analysis–introduced by Black and Scholes [4], and extended by Merton [56] and Heston [26] – ensure continued relevance [31]. However, liquidity shocks, which deviate from the traditional Black-Scholes framework's assumptions, limit the market's completeness[1] by reducing the liquidity of stocks [54, 74].

Partial differential equations (PDEs) have been crucial in financial analysis due to their ability to model complex dynamics and temporal interconnections of financial derivatives [18]. PDE-based models help understand and predict financial phenomena such as risk management, portfolio optimization, option pricing, and bond pricing [12, 25, 72]. In particular, solving the PDEs using Lie symmetry analysis provides structured solutions to differential equations, often difficult to achieve with traditional numerical methods [37]. This method simplifies complex parabolic systems and reveals solutions that provide deeper insights into option price behavior during liquidity shocks [54, 63].

Liquidity shocks in European options cause difficulties in pricing, and the thesis aims to provide treatment. We use the Lie symmetry analysis to investigate the model's behavior of option prices in

[1] a *complete* trading market has liquidity, transparency, diverse participants, regulation, and efficient price discovery, ensuring smooth and fair operations

turbulent market conditions using a parabolic system of partial differential equations. The parabolic system, belonging to a class of heat equations, is proficient in encapsulating the temporal and spatial intricacies of option pricing dynamics under the influence of liquidity perturbations [63].

The Introduction lays the foundation for an in-depth study, starting with a brief overview of European option pricing and the effects of liquidity shocks. Next, we explore the mathematical formulation of the parabolic system of PDEs, setting the stage for applying Lie symmetry analysis. We will highlight the historical evolution of this analytical method and its pivotal role in modern mathematical finance.

Our aim is to close the knowledge gap between financial theory and practice by providing a solid framework for comprehending and forecasting the behavior of European options in the presence of a liquidity crunch. With this work, we hope to further the ongoing discussion between financial engineers and mathematicians, promoting a cooperative effort to improve and refine the models that serve as the foundation for our financial systems.

1.1 Background

1.1.1 Financial Derivatives and Options Pricing

Financial derivatives are complex tools that are essential for contemporary financial markets because they can be used for both hedging risk and speculation [29, 31, 48]. They come in many different forms, such as forward contracts, swaps, futures, and most famously options contracts, and their value is dependent on an underlying asset, index, or rate [31, 33, 49]. Options are particularly significant because they give the holder the option, but not the obligation, to buy or sell the underlying asset within a predetermined period and at a predetermined price [31]. Derivatives are essential for the strategic management of financial risk, making this idea fundamental to modern financial theory and practice [21, 31].

Historical Development

The French mathematician Louis Bachelier [2] established the theoretical foundation for option pric-

ing in 1900 by proposing an option pricing model predicated on the notion that stock prices move in a Brownian motion. However, the field did not take off until Fischer Black and Myron Scholes [4] introduced the Black-Scholes model in 1973. This model became the cornerstone of contemporary finance theory and offered a novel approach to determining the fair pricing of options. By adding new presumptions and developing new explicit formulas for various options, such as call, put, and exotic options such as down-and-out options, Robert Merton [56] expanded the application of the Black-Scholes theory.

Mathematical Models in Option Pricing

The theory of option pricing revolves around the partial differential equation known as the Black-Scholes equation [30]. It is expressed as:

$$\frac{\partial u}{\partial t} + \frac{1}{2}x^2\sigma^2\frac{\partial^2 u}{\partial x^2} + r(x\frac{\partial u}{\partial t} - u) = 0 \tag{1.1}$$

where u denotes the option price, x represents the stock price, σ signifies the volatility, and r stands for the risk-free interest rate, employed to ascertain the theoretical valuation of European options [4]. With the development of the first bond pricing model in 1977, Oldrich Vasicek's [78] groundbreaking research transformed the area of financial modeling. Vasicek's model provided a probabilistic framework to account for the random nature of interest rate movements by cleverly using a stochastic differential equation to describe the term structure of interest rates. This groundbreaking model not only served as a template for subsequent one-factor models but also offered a method to estimate bond prices. Vasicek's approach to describing the dynamic evolution of interest rates over time has been expanded upon by later models, such as the Cox-Ingersoll-Ross (CIR) model [10] and the Longstaff model [51]. These models have been crucial in the valuation of a variety of interest rate derivatives. They help assess and manage the financial risk related to interest rate variations by incorporating variables like volatility, and mean reversion.

Advancements in Option Pricing Models

After the Black-Scholes and Vasicek models were introduced, the option pricing landscape experienced a dramatic change. These early models provided the foundation for understanding how options

might be valued in a market where stochastic factors are at play. However, they did have certain drawbacks, mainly related to their presumptions of constant interest rates and volatility [61].

Researchers have developed more advanced models that provide a closer reflection of market behaviors to overcome these limitations. Among them is the Cox-Ingersoll-Ross (CIR) model [10]. Mean reversion, which postulates that interest rates move around a long-term average value rather than staying constant, casts doubt on the notion that interest rates are constant. This model offers a more realistic framework for pricing interest rate derivatives and is especially helpful in capturing the behavior of interest rates over time.

The Hull-White model [30], which applies stochastic volatility to option pricing, is another noteworthy advancement. In contrast to the Black-Scholes model's constant volatility assumption, the Hull-White model recognizes that volatility is unpredictable and typically follows a mean-reverting process. This implies that, while volatility may occasionally stray from its long-term average, it will eventually return to it. The Hull-White model provides a more dynamic approach to the pricing of options, particularly those with longer maturities, where volatility changes can have a more pronounced impact, taking into account this stochastic nature of volatility.

Liquidity shocks and market imperfections have been the focus of recent advances in option pricing [54, 63]. For example, taking liquidity risks into account results in more realistic pricing models that take into account how trading volume and market depth affect option prices.

1.1.2 Liquidity Shocks in Financial Markets

Sudden and unforeseen shifts in the liquidity of financial markets are known as liquidity shocks, and they can have a big influence on trading volumes, asset prices, and the stability of the market as a whole [54]. Investigating how liquidity shocks affect European option pricing, Michael Ludkovski and Qunying Shen et al. in [54] offer a comprehensive model that takes these shocks into account when determining option prices.

The idea that agents have unrestricted trading freedom and that markets are always liquid is a cornerstone of the conventional option pricing theory [49, 70]. However, as Longstaff et al. [52] point

out, this is usually not the case in real financial markets. Although there is now a consensus on the definition of liquidity risk, research on the impact of liquidity risk on derivative pricing is still in its early stages [49, 53, 63]. From a particular academic perspective [70], the fundamental source of liquidity risk is the impact exerted by traders' activities on asset prices. For example, Liu and Yong [49] used the assumption that a trader's activity affects the underlying price to derive a generalized Black-Scholes pricing equation.

Several strategies have been developed in recent years to address market illiquidity within optimal investment frameworks. Gassiat et al. [16] and Ludkovski and Min [53] examined how illiquidity affected the optimal consumption plans. Ludkovski and Shen [54] extended the analysis to the problem of pricing derivative securities. They presented a framework in which a continuous-time Markov chain controls the market's alternating states of liquidity and illiquidity. By combining a parabolic PDE and an ordinary differential equation (ODE), this method yields an integro-differential equation that takes into account changes in the liquidity state and how they affect option prices.

Looking at the well-posedness and comparison principles of the resulting integro-differential equations, Gyulov and Vulkov [22] expanded on this model. For the pricing model to be accurate and reliable, their research guarantees that the solutions to these equations are distinct and stable. These fundamental characteristics laid the groundwork for additional analytical and numerical investigation of the model. The literature is further expanded by Gyulov and Koleva's work on the "penalty method for indifference pricing of American options in liquidity-switching markets" [23], which tackles the nonlinear complementarity problem (NLCP) that these models inherently have. They apply the interior penalty method, which guarantees strict satisfaction of the early exercise constraint of American options, and give comprehensive proof of the convergence, boundedness, and uniqueness of their numerical solutions.

Theoretical models are validated with market data by empirical studies, such as those conducted by Liu and Yong [49]. The aforementioned studies underscore the noteworthy influence of liquidity on option prices and offer pragmatic perspectives on the application of these models in trading and risk mitigation. It is further demonstrated by Avellaneda and Stoikov's [1] research on high-frequency trading strategies how useful it is to include liquidity considerations in option pricing models.

1.1.3 Lie Symmetry Analysis in Mathematical Finance

Lie symmetry analysis offers a strong framework for the identification and classification of solutions to differential equations [45], which has its roots in the work of the Norwegian mathematician Sophus Lie (Figure 1.1). Lie's techniques provide methodical ways to find precise, frequently unique solutions by taking advantage of the differential equations' invariance properties under continuous transformation groups. Numerous fields, including mathematical finance, have found significant uses for this analytical toolkit.

Figure 1.1: Sophus Lie (Nordfjordeid, 17 December 1842—Christiania, 18 February 1899).

The Lie symmetry approach is quite computational and analytical. The system-at-a-glance expands and integrates ad-hoc methods to build explicit solutions for differential equations. To find symmetries defined by differential equations and to create invariant solutions that come from the symmetries, the focus is on explicit computational procedures [5, 32]. The ability to lower an ODE or PDE's order by one if it is invariant under one parameter is a prominent characteristic of symmetry analysis under Lie group point transformations [67]. Lie group analysis established itself as an effective method of solving nonlinear differential equations analytically. Specifically, Sophus Lie provided the first general solution to the problem of classifying partial differential equations based on their symmetries [34].

Numerous researchers have worked on different families of differential equations since then. The results of their efforts have been depicted in several exceptional literary works by, among others, Ibragimov [17, 34], Hydon [32], Bluman [5], etc.

Table 1.1 above depicts the application of Lie Symmetry Analysis in Mathematical Finance and Table

Table 1.1: Applications of Lie Symmetry Analysis in Mathematical Finance

Application	**Description**
Bond Pricing	Lie symmetry analysis is used to simplify the bond pricing PDE, often leading to closed-form solutions. For example, the bond price equation: $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda\sigma)\frac{\partial V}{\partial r} - rV = 0$$ can be analyzed using symmetries to reduce its complexity and find exact solutions for bond prices.
Option Pricing	The Black-Scholes equation for option pricing can be transformed using Lie symmetries to obtain simpler forms, facilitating the derivation of pricing formulas for various options. The equation: $$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$ is often simplified by using symmetry methods.
Interest Rate Models	Models like Vasicek and Cox-Ingersoll-Ross (CIR), described by stochastic differential equations (SDEs), benefit from symmetry analysis. These models: $$\frac{\partial u}{\partial t} + \alpha x \frac{\partial^2 u}{\partial x^2} + \kappa(\theta - x)\frac{\partial u}{\partial x} + \gamma x u = 0,$$ are simplified to provide deeper insights and more efficient computational techniques, one used to solve them.

1.2 summarizes the relevant published literature with their focused perspectives.

Table 1.2: Recurring Research Themes Over the Years

Year	Authors	Brief Research Theme
1880	Sophus Lie [45]	To develop DE invariant of the continuous groups transformation
1872 - 1899	Sophus Lie [45, 46]	Introduced and applied the symmetry analysis of DEs
1978	Ovsiannikov [68]	Systematically exploited the methods of DEs symmetry analysis in the explicit solutions construction for multifaceted problems
1990	Bluman [6]	Proposed a reduction algorithm for ODEs by admitting a solvable Lie group
1993	Olver [67]	Introduced the use of Lie groups to differential equations
1998	Gazizov and Ibragimov [17]	Introduced Lie Symmetry analysis in the field of Mathematical finance to solve the Black-Scholes equation and transform the BS equation into heat equations
2010	Caister, O'hara and Govinder [7]	Lie symmetry analysis was used to obtain Asian option solutions that satisfy the terminal conditions
2011	Liu and Wang [50]	Solved the BS equation with dividend yield using the symmetry approach
2013	O'Hara, Sophocleous and Leach [69]	Used Lie Symmetry analysis to solve a model that exercises a barrier option
2015	Okelola, Govinder, and O'Hara [66]	Studied a partial differential equation associated with the pricing of power options with time-dependent parameters
2018	Khalique and Motsepa [39]	Investigated the application of Lie symmetry analysis to the Vasicek pricing equation
2019	Kaibe and O'Hara [38]	Studied the zero-coupon pricing equation to determine the symmetries point to reduce the equation into the ODE and hence, the solutions were obtained
2021	K. C. Bosiu [37]	Solved Partial Differential Equations associated with Mathematics of Finance
2022	Tang, Chong, and Kee[77]	Derived invariant solutions for the generalized CIR model and constructed optimal subalgebras

1.2 Research Objectives

The principal objective of this research is to investigate and analyze a parabolic system that models the valuation of European options under liquidity shocks. Liquidity shocks alter market completeness

by affecting the availability of tradable assets, and the model studied here captures both the asset price diffusion and the state of market liquidity.

The system under investigation is adapted from liquidity-switching models in the literature Ludkovski and Shen [54] and consists of a degenerate parabolic partial differential equation coupled with a nonlinear first-order ordinary differential equation in time:

$$\begin{aligned} &u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} + a e^u e^{-v} - b = 0, \\ &v_t + c e^{v-u} - c = 0. \end{aligned} \tag{1.2}$$

where $a = \nu_{01}$, $b = d_0 + \nu_{01}$, $c = \nu_{10}$, and $u(t, x)$ and $v(x, t)$ denote the option value functions in two market liquidity states, σ is the volatility of the underlying asset, and x is the asset price. The terminal conditions for a European call option are given by:

$$u(0, x) = u_0(x) = \gamma h(x), \quad v(0, x) = v_0(x) = \gamma h(x),$$

with the payoff function:

$$h(x) = \max(S - K, 0).$$

Within this context, the specific objectives of the research are:

1. **Lie Symmetry Analysis on the Model:**
    - Perform a Lie symmetry analysis to obtain the determining equations. To do this, we leverage the power of the SYM package [13] in conjunction with the Mathematica 12.1 [79] capabilities.
2. **Solution of Over-Determined Systems:**
    - Solve the over-determined systems of equations to obtain the Lie symmetry generators for the model.
3. **Construction of Lie Point Symmetries:**

- Construct the Lie point symmetries for the model based on the obtained symmetry generators.

4. **Commutation:**

   - Establish the closure of symmetries under commutation through the construction of a tabulation of Lie algebras.
   - Construct the optimal system of subalgebras.

5. **Symmetry Reductions:**

   - Perform symmetry reductions using the optimal system of sub-algebras to reduce the partial differential equations to ordinary differential equations.

6. **Generation of Invariant Solutions:**

   - Generate a family of exact closed-form invariant solutions for the model and analyze them using numerical simulations.

1.3 Thesis Outline

This thesis is structured as follows:

Chapter 2: Preliminaries This chapter provides the necessary mathematical background, including a review of the fundamentals of option pricing, parabolic PDEs, and Lie symmetry theory.

Chapter 3: Model Framework
We derive the parabolic system for European option pricing under liquidity shocks.

Chapter 4: Lie Symmetry Analysis Applied to the Coupled System in the Modeling of European Options It is the core of the thesis, this chapter applies the Lie symmetry methods to the formulated PDE system. We determine the symmetries, perform symmetry reductions, and obtain exact solutions. We finally make simulations of the exact solutions.

Chapter 5: Discussions and Conclusion

This last chapter provides an overview of the research, highlights its contributions, and makes recommendations for future directions. By the time this thesis concludes, we hope to have established a thorough framework—using the potent instrument of Lie symmetry analysis—for comprehending and resolving the difficulties associated with option pricing when liquidity shocks are present.

2

Preliminary

This chapter provides the foundational mathematical framework essential for the analysis and developments presented in this thesis. It is set in a way that connects mathematics and finance interdependently. An overview of notations and terminologies of stochastic calculus and financial derivatives is given, and they will be employed throughout this study. Partial differential equations (PDEs) are then discussed, emphasising how they can be used to model dynamic systems and their relation to the pricing of derivatives. We end the chapter by stating concepts and main theorems of Lie symmetry method for PDEs. In particular, the Lie symmetry method for computing the infinitesimal symmetries and the invariant solutions of the PDEs. We begin with the financial derivatives.

2.1 Derivatives in Financial Mathematics

Derivatives are financial products that have grown in importance in financial markets [31]. This is because derivatives have multiple uses [31, 64]. For example, they are frequently included in bond offerings, and some businesses also include them in the chief executive officer compensation schemes. Above all, they can shift the mortgage risk from the original lender to a different investor [33, 59]. To

provide further insight into these, particularly the pricing of European options, which is central to this thesis, we give an overview of the mathematical terminology employed throughout. We define terms such as random variables, stochastic differential equations, filtering, Brownian motion, and stochastic processes.

2.1.1 Random Variables and Stochastic Processes

The triple $(\Omega, \mathcal{F}, \mathbb{P})$, comprising of a set Ω, a σ - field $\mathcal{F}$ of subsets of Ω, and a probability $\mathbb{P}$ on $(\Omega, \mathcal{F})$, is called a probability space. The collection $\mathcal{F}$ is a σ - field, that is, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ are closed under the operations of countable union and taking compliments. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ satisfying

1. $\mathbb{P}(\emptyset) = 0$,
2. $\mathbb{P}(\Omega) = 1$,
3. if $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, in that $A_i \cap A_j = \emptyset$ for all pairs i, j satisfying $i \neq j$, then
$$\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}\left(A_i\right).$$

Next, we will explain the theory related to our subject, *Random Variables and Stochastic Processes*.

Definition 2.1. *A random variable is a mathematical representation of an uncertain quantity in a probability space. It assigns a numerical value to each possible outcome of a random experiment.*

Random variables can be discrete, taking on a countable set of values, or continuous, taking on values from a constant range. They play a fundamental role in probability theory and statistics, allowing us to quantify and analyze the uncertain nature of events and phenomena.

Definition 2.2. *A stochastic process is a collection of random variables indexed by time or another parameter, representing the evolution of a system over time in a probabilistic manner.*

The stochastic process provide a mathematical framework for modeling and analyzing random phenomena that change dynamically. Each random variable in a stochastic process represents the system's state at a specific time. Stochastic processes can be classified as discrete-time or continuous-time, depending on whether the index set is discrete or continuous. They find extensive applications in various fields such as finance, engineering, physics, and biology, enabling studying of dynamic systems affected by random influences.

Definition 2.3. *[30] The cumulative distribution function of a random variable X is the function $F : \Re \longrightarrow [0, 1]$ given by $F(x) = \mathbb{P}(X \leq x)$.*

Definition 2.4. *[31] Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a σ-algebra $\mathcal{F}_t$. Assume further that $\mathcal{F}_s \subset \mathcal{F}_t$ for all $0 \leq s < t < \infty$ and $\mathcal{F} = \bigcup_{t\geq 0} \mathcal{F}_t$. Then we call the collection $\mathcal{F}_t$ of σ-algebras a filtration, which $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ is called a filtered probability space. $\mathcal{F}_t$ is taken to be the set of information available to the observer (e.g. the investor or the bank manager) up to time t. More specifically, $\{\mathcal{F}_t\}_{t>0}$ is considered to be the flow of information over certain time and it is assumed that the bank does not lose information as time passes (hence why we say $\mathcal{F}_s \subset \mathcal{F}_t$ for $s < t$).*

Definition 2.5. *[31] A real-valued stochastic process is an indexed family of real-valued functions $\{X_t\}_{t\geq 0}$ that Ω. $\{X_t\}_{t\geq 0}$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if X_s is $\mathcal{F}_t$-measurable for each $t \geq s$.*

2.1.2 Brownian Motion

This is the process $W = \{W(t) : t \geq 0\}$, indexed by continuous time and taking values in the real line $\mathbb{R}$, which is time-homogeneous with independent increments, and with the vital extra property that $W(t)$ has the normal distribution with mean 0 and variance $\sigma^2 t$ for some constant σ^2. Brownian motion is the cornerstone of the modern theory of random processes [20]. In 1827, Robert Brown, a Scottish botanist, discovered the phenomenon known as Brownian motion. While observing pollen grains suspended in water under a microscope, he noticed that the tiny particles within the pollen moved in a continuous, jittery manner [44]. It was later discovered that such irregular motion comes from a huge number of collisions of the suspended pollen grains with the liquid molecules. Norbert

Wiener presented a mathematical model for this motion based on the theory of stochastic processes [36]. The position of a particle at each time $t \geq 0$ is a three-dimensional random vector W_t.

Definition 2.6. *[31] A real-valued stochastic process $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion (or a $\mathbb{P}$-Wiener process) if for some real constant σ, under $\mathbb{P}$,*

1. *for each $s \geq 0$ and $t > 0$ the random variable $W_{t+s} - W_s$ has the normal distribution with mean zero and variance $\sigma^2 t$,*
2. *for each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$, the random variables $\left\{W_{t_r} - W_{t_{r-1}}\right\}$ are independent,*
3. *$W_0 = 0$,*
4. *W_t is continuous in $t \geq 0$.*

2.1.3 Stochastic Integration

Brownian motion [35], has played a vital role in stochastic integration of risky asset prices and their modeling. Wiener built upon Brown's discovery by deriving and proving many properties associated with the paths of Brownian motion [36]. The following two key properties relate the stochastic integration: (1) the paths of Brownian motion have a nonzero finite quadratic variation, such that on an interval (s, t), the quadratic variation is $(t - s)$, and (2) the paths of Brownian motion have an infinite variation on compact time intervals, almost surely. Processes used to model stock prices are usually functions of one or more Brownian motions. In this regard, suppose that the stock price is of the form $S_t = f(t, Z_t)$. Using Taylor's theorem, we can write

$$f(t + \delta t, Z_{t+\delta t}) - f(t, Z_t) = \delta t \bar{f}(t, Z_t) + O\left(\delta t^2\right) + (Z_{t+\sigma_t} - Z_t) f'(t, Z_i) + \frac{1}{2!}(Z_{t+\delta t} - Z_t)^2 f''(t, Z_t) + \ldots$$

where the notation $\bar{f}, f'$ and f'' must be interpreted as $\bar{f}(t,x) = \frac{\partial f}{\partial t}(t,x)$, $f'(t,x) = \frac{\partial f}{\partial x}(t,x)$ and $f''(t,x) = \frac{\partial^2 f}{\partial x^2}(t,x)$. The dynamics of a stock price are commonly modeled by way of a stochastic

differential equation as follows (see for example [27]):

$$dS_t = \bar{f}\,(t, Z_t)\,dt + f'\,(Z_t)\,dZ_t + \frac{1}{2} f''\,(Z_t)\,dt.$$

The differential equation above is also resembled in integrated form as follows,

$$S_t = S_0 + \int_0^t \bar{f}\,(s, Z_s)\,ds + \int_0^t f'\,(Z_s)\,dZ_s + \int_0^t \frac{1}{2} f''\,(Z_s)\,ds. \tag{2.1}$$

Definition 2.7. *Itô Process (or stochastic integral) is a stochastic process X_t on $(\Omega, \mathcal{F}, \mathbb{P})$ of the form [71]*

$$X_t = X_0 + \int_0^t a\,(X_t, t)\,ds + \int_0^t b\,(X_t, t)\,dZ_s.$$

If X_t is an Ito process of the form (2.1) the equation is sometimes written in the shorter differential form

$$dX_t = a\,(X_t, t)\,dt + b\,(X_t, t)\,dZ_t,$$

where $a\,(X_t, t)$ is the drift rate, $b\,(X_t, t)$ is the variance rate or diffusion and Z_t is a standard Wiener process.

Definition 2.8 (Itô Formula)**.** *[64] Let X_t be an Ito process given by*

$$dX_t = a\,dt + b\,dZ_t, \tag{2.2}$$

and $g(t, x) \in C^2([0, \infty) \times \mathbb{R})$, (i.e, g is twice continuously differentiable on $[0, \infty) \times \mathbb{R}$) , then

$$Y_t = g\,(t, X_t)\,,$$

is again an Itô process, and

$$dY_t = \frac{\partial g}{\partial t}\,(t, X_t)\,dt + \frac{\partial g}{\partial x}\,(t, X_t)\,dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\,(t, X_t) \cdot (dX_t)^2\,,$$

where differentials are multiplied according to the rules

$$dt \cdot dt = dt \cdot dZ_t = dZ_t \cdot dt = 0, \quad dZ_t \cdot dZ_t = dt.$$

The following example shows how the values of the stochastic integrals can be determined using the Itô Formula.

Example 2.1.1. *[64] Suppose that we choose $X_t = B_t$ and $f(t,x) = x^2$ where B_t is the Brownian motion. The first and second derivatives of $f(t,x)$ are respectively $f_t^{'}(t,x) = 0$, $f_x^{'}(t,x) = 2x$ and $f_{xx}^{'}(t,x) = 2$. So, $Y_t = f(t,B_t) = B_t^2$. The Itô formula will be*

$$\begin{aligned} dY_i &= \frac{\partial f}{\partial t}(t,X_i)dt + \frac{\partial f}{\partial x}(t,X_i)dX_i + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,X_i)\cdot(dX_i)^2, \\ &= 0dt + 2B_i dB_i + (dB_i)^2, \\ &= 2B_i dB_i + dt. \end{aligned}$$

Consequently,

$$d(B_i^2) = 2B_i dB_i + dt.$$

For $s \in [0,t]$, the integral form will look like this:

$$B_t^2 = 2\int_0^t B_s dB_s + t.$$

We determine that

$$\int_0^t B_s dB_s = \frac{B_t^2 - t}{2},$$

based on the previous representation. We observe that there is a connection between the stochastic process and a second-order partial differential equation using the Itô Formula. Typically, the generators listed below are used to define this relationship.

Definition 2.9. *[64] Let $\{X_t\}$ be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The (infinitesimal) generator L of $\{X_t\}$ is defined by*

$$Lf(x) = \lim_{t\to 0}\frac{\mathbb{E}^x[f(X_t)] - f(x)}{t}, \quad x \in \mathbb{R}^n. \tag{2.3}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_L(x)$, while $\mathcal{D}_L$ denotes the set of functions for which the limit exists for all $x \in \mathbb{R}^n$.

The deduction of the generator formula L for an Itô diffusion process can be found in [27], but the principal idea is to consider the expectation $E^x[f(X_t)]$ in (2.3) can be calculated by

$$\begin{aligned} E^x[f(X_t)] = f(x) + \sum_{ik} E^x\left[\sigma_{ik}\frac{\partial f}{\partial x_i}(X)dB_k\right] \\ + E^x\left[\int_0^t \left(\sum_i b_i \frac{\partial f}{\partial x_i}(X) + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(X)\right) ds\right]. \end{aligned} \tag{2.4}$$

We assume that f is a bounded Borel function so that the limit of the second term in (2.4) vanishes and any Itô diffusion process X_i in R^n will always be associated with the generator L given by

$$L_f = \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}(\sigma^T\sigma)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}; \quad f = f(x) \in C_0^2(\mathbb{R}^n). \tag{2.5}$$

2.1.4 Financial Derivatives

This subsection provides an overview of financial tools used to manage financial risks. It covers the concepts used for the evaluation of financial derivatives, which will be applied during this study. More details about financial derivatives can be found in [4, 18, 21, 29, 30, 31, 33, 59, 64] and the references therein.

Products and Fundamentals

To survive and thrive, companies will always face certain risks. These risks may include equity risks such as interest rates, exchange rates, and commodity prices, or less conventional risks such as weather, energy prices, and insurance. Generally, companies lack the expertise to predict these variables and therefore prefer to hedge (taking an opposite position in a related asset) the associated risks as a solution. The idea behind hedging is to avoid surprises and focus solely on production. Furthermore, there is a trade-off between risk and return, as higher expected returns can only be

achieved by taking higher risks. Risk management is a key tool to understand the current portfolio of risks and to plan for future risks. In the financial market, various financial tools known as contingent claims can be utilized to manage financial risk.

Definition 2.10. *[64] Contingent claim-T or T-claim is a contract that pays to the holder a stochastic amount X at time T. The random variable X is $\mathcal{F}_T$-measurable and T is called the exercise time of the contingent claim or maturity of the contract.*

The shared feature of contingent claims is that they are characterized by the type of financial goods to be traded and the underlying asset. They can define certain financial markets, including:

- Stock markets: familiar notion of stock exchange markets such as New York, London, Tokyo, Milan;
- Bond markets: products with fixed returns, usually issued by the central banks;
- Currency markets or foreign exchange: currencies and their prices are determined;
- Commodities markets: commodities prices such as oil, and gold, are fixed;
- Future, forward, swaps, options markets: derivatives products based on one or more other underlying products typically of the previous markets.

One may also take into account sub-markets inside the futures and options markets that are based on the particular underlying asset, such as weather, oil, or power derivatives. The weather, oil, and power are the fundamental assets, in that order. Contracts related to financial goods are known as derivatives. These contracts can be applied to standard, plain-vanilla, structured, or exotic items. The last group is made to specifically address the demands of the company treasurer. Options, futures, swaps, and forward contracts are the most frequently traded basic products.

Forward Contract

A forward contract is an agreement between two parties to purchase an asset at a future date for a predetermined price. One party takes a long position, agreeing to buy the asset at a specific future

date for a specified price, while the other party takes a short position, agreeing to sell the asset on the same date for the same price. These contracts are typically traded in the over-the-counter (OTC) market.

The OTC market is not a formalised market. The contract terms do not need to adhere to those specified by an exchange, and market participants are free to negotiate any mutually agreeable deal. In contrast, exchange-traded markets are organized markets where contracts and trading among participants are clearly defined and regulated by the exchange, ensuring that agreed trades are honored.

Futures Contracts

Similar to forward contracts, futures contracts are agreements to purchase an asset at a future date. However, unlike forwards, futures are traded on an exchange, meaning the contracts are standardised. The exchange specifies the amount of the underlying asset, the delivery date, and other terms. Futures contracts can also be closed before the delivery date.

Swaps

A swap is an agreement between two companies to exchange cash flows in the future. The agreement specifies the dates on which cash flows will be exchanged and the calculation method. Forward contracts can be seen as a type of swap, as they lead to a single exchange of cash flows on a future date. In contrast, swaps involve multiple cash flow exchanges over several future dates.

Options

Options are financial instruments that can be traded on both formal exchanges and in the OTC markets. They come in two primary types: *call options* and *put options*. A call option grants the holder the right to purchase the underlying asset at a specific price on a specific date, while a put option grants the right to sell the underlying asset at a specific price on a specific date. The price specified in the

contract is known as the *exercise price* or *strike price*, and the specified date is called the *expiration date* or *maturity*.

Depending on the exercise date, options can be categorized into *American options* and *European options*. American options can be exercised at any point up to the expiration date, whereas European options can only be exercised on the expiration date itself. Unlike forward or futures contracts, options provide the right but not the obligation to buy or sell the underlying asset, and there is a cost associated with taking a position in these contracts.

On the exercise date, the option can be in one of three states: *at-the-money*, *out-of-the-money*, or *in-the-money*. An at-the-money option is when the strike price is close to the underlying asset's price. An out-of-the-money option is when a call option's strike price is above the underlying asset's price or when a put option's strike price is below the underlying asset's price. Conversely, an in-the-money option is when a call option's strike price is below the underlying asset's price or when a put option's strike price is above the underlying asset's price.

Payoff Functions for European Options

European call options have a payoff function denoted by $f(x) = \max(x - K, 0)$, where K represents the strike price of the option. For European put options, the payoff function is given by $f(x) = \max(K - x, 0)$.

To elaborate, the payoff of a European call option depends on whether the final stock price S_T exceeds the strike price K at expiration. Specifically, if S_T is greater than K, the payoff is the difference $S_T - K$; otherwise, the payoff is zero. This can be mathematically expressed as:

$$f(S_T) = \max(S_T - K, 0).$$

This indicates that the holder of the call option benefits only when the stock price at expiration S_T is above the strike price K.

Similarly, for a European put option, the payoff is determined by whether the strike price K exceeds

the final stock price S_T at expiration. If K is greater than S_T, the payoff is $K - S_T$; otherwise, the payoff is zero. This can be represented as:

$$f(S_T) = \max(K - S_T, 0).$$

This means that the holder of the put option benefits only when the stock price at expiration S_T is below the strike price K.

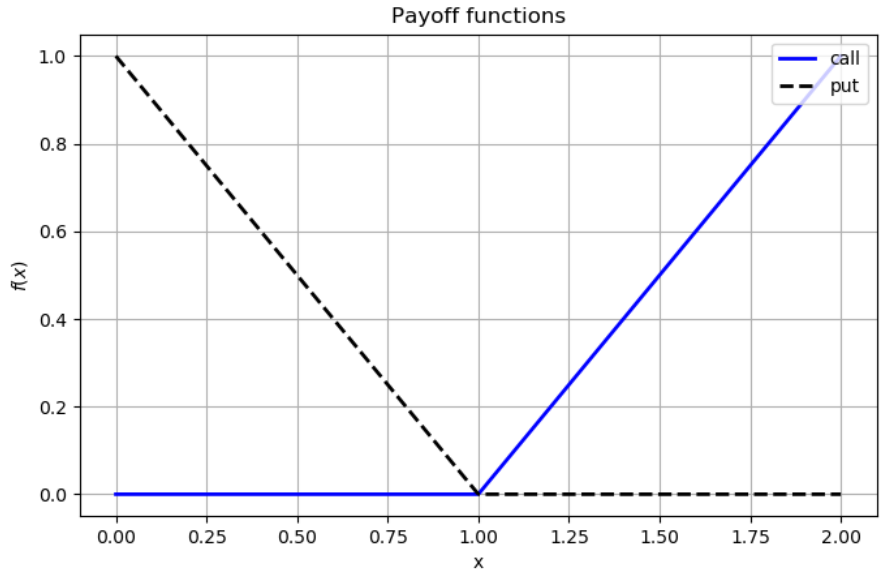


Figure 2.1: Payoff functions for European call and put options. The call option payoff is shown in blue, and the put option payoff is shown in black.

2.2 Partial Differential Equations (PDEs) and Their Classifications

In applied mathematics and mathematical physics, PDEs are fundamental tools for modeling various phenomena. The most general second-order linear PDE in two independent variables x and y is

expressed as:

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x} + E(x,y)\frac{\partial u}{\partial y} + F(x,y)u = G(x,y), \tag{2.6}$$

where A, B, C, D, E, and F are functions of x and y, and $u(x,y)$ is the unknown function.

The classification of these PDEs depends on the discriminant $\Delta = B^2 - 4AC$:

- **Elliptic PDEs**: When $\Delta < 0$. These equations, such as the Laplace equation, typically describe steady-state processes.
- **Parabolic PDEs**: When $\Delta = 0$. These include the heat equation, used to model time-dependent processes like heat conduction.
- **Hyperbolic PDEs**: When $\Delta > 0$. These equations, like the wave equation, are associated with wave propagation and dynamic systems.

To solve these PDEs, the method of characteristics is often employed. This technique simplifies the PDE by transforming it into a set of ODEs, making it easier to solve. Additionally, the Lie symmetry technique will be employed, particularly for solving the parabolic PDE, it transforms the PDE into simpler ODEs.

2.3 Parabolic Partial Differential Equations

Parabolic partial differential equations (PDEs) are a basic class of equations in applied mathematics and mathematical physics. They are vital resources for simulating a wide range of time-dependent processes, such as the pricing of financial derivatives, particle diffusion, and heat conduction, among others [15, 58]. The unique aspect of parabolic PDEs is their ability to depict the evolution of systems, including the propagation and alteration of waves and patterns.

An example on a second-order parabolic PDE in two variables, x and t, which typically represent temporal and spatial dimensions, respectively, can be expressed in the following form:

$$\frac{\partial u}{\partial t} - a\frac{\partial^2 u}{\partial x^2} = f(x,t). \tag{2.7}$$

The unknown function we are searching for is indicated by the symbol $u(x,t)$. The source term, $f(x,t)$, can indicate outside forces or influences, and the coefficient, a, is a constant that describes the characteristics of the medium or the system [43].

It is the nature of their coefficients and the boundary and initial conditions that distinguish parabolic PDEs from other PDE categories, like elliptic and hyperbolic PDEs. The Laplace equation serves as an example of an elliptic PDE, which is usually used for steady-state problems in which time is not a factor. On the other hand, dynamic systems where wave propagation plays a crucial role are modeled using hyperbolic PDEs, such as the wave equation [11].

A fundamental property of the parabolic PDEs is their smoothing effect on solutions. The smoothing effect of solutions to parabolic partial differential equations (PDEs) refers to the phenomenon where the solutions become progressively smoother over time, regardless of the initial conditions. This characteristic is notable because it implies that any irregularities or discontinuities in the initial data diminish as the system evolves. This property reflects the behavior of physical systems as they approach equilibrium and goes beyond simple mathematical curiosity. The physical phenomena they represent, such as the gradual uniformity of temperature distribution resulting from heat diffusion in a solid object, are mirrored by this smoothing property.

2.3.1 Applications of Parabolic PDEs

In financial mathematics, especially in option pricing models, parabolic PDEs are essential. A key tool in derivative pricing, the Black-Scholes equation is a parabolic PDE that illustrates how option prices change over time. These formulas aid in risk assessment, asset price modeling, and taking into consideration a number of market variables, including volatility and liquidity shocks. In this situation, we can effectively hedge financial risks and ascertain fair option prices by solving parabolic

PDEs [42]. Financial derivatives pricing is modeled in the field of finance using parabolic PDEs [34].

Parabolic PDEs are also useful in environmental science, especially when simulating the dispersion of contaminants in air or water [3]. By predicting the spread of contaminants, these equations help in the development of efficient pollution control and remediation strategies.

2.3.2 Analytical and Numerical Solutions

It can be difficult to solve parabolic PDEs, and a number of techniques are used based on the boundary conditions and equation complexity. For simpler equations with well-defined boundary conditions, analytical solutions are feasible. These solutions yield exact expressions for the unknown function $u(x, t)$, providing deep insights into the behavior of the system [19].

Numerical techniques such as spectral, finite element, and finite difference methods are applied in these situations [49, 63]. By discretizing the PDE and the domain, these methods reduce the continuous problem to an algebraic system of equations that can be resolved by computer programs [76].

2.3.3 Initial and Boundary Conditions

When figuring out the solution to a parabolic PDE, the initial and boundary conditions are very important. The system's state at the start of the observation period is specified by the initial condition. For example, the initial temperature distribution across the material should be known to solve the heat equation [76].

The behavior of the solution at the domain's boundaries is specified by boundary conditions. Boundary conditions come in two common forms: Neumann boundary conditions, which fix the value of the solution's derivative at the boundary, and Dirichlet boundary conditions, which fix the value of the solution at the boundary. Moreover, a variety of applications employ Robin boundary conditions, which combine Neumann and Dirichlet conditions [11, 47, 65].

2.4 Lie Point Symmetries of Differential Equations

One effective technique for finding analytical solutions to differential equations is Lie symmetry analysis and we are going to use it in our work. The concept of differential equation invariance, which enables the solution of differential equations using Lie groups, is fundamental to this algorithmic process, which has its roots in Lie's research. There are many worthwhile manuscripts [6, 8, 32, 34, 37, 40, 60, 67] in the literature that provide a thorough introduction to the topic. The above-mentioned references serve as the foundation for this section's definitions, theorems and annotations.

2.4.1 One Parameter Group of Transformations

Definition 2.11. *[60] A k-th order* $(k \geq 1)$ *system* E *of* s *differential equations is defined by*

$$E^{\sigma}(x, u, u_{(1)}, ..., u_{(k)}) = 0, \quad \sigma = 1, ..., s. \tag{2.8}$$

where $u \equiv (u^1, u^2, ..., u^m)$ *is dependent vector,* $x \equiv (x^1, x^2, ..., x^n)$ *is the independent vector and* $u_{(1)}, u_{(2)}, ..., u_{(k)}$ *are respectively the collection of all first, second, up to k-th-order partial derivatives. In expanded form*

$$u_{(1)} = \{u_i^{\alpha}\}, \quad , u_{(2)} = \{u_{ij}^{\alpha}\}, \quad , u_{(k)} = \{u_{i1}^{\alpha}\} \tag{2.9}$$

where $\alpha = 1, ..., m; \quad i, j, i_1 \cdots i_k = 1, ..., n.$

The equations (2.8) have the highest possible order, which is k. In numerous cases, the number of equations (s) is equal to the number of unknowns (m). It should be noted that if $n = 1$, the equations in (2.8) turn into a system of ODEs, while if $n > 1$, it becomes a system of PDEs [60].

Definition 2.12. *[67] A symmetry transformation of the system* (2.8) *is an invertible transformation of the dependent variables* u *and independent variables* x*,namely*

$$\bar{x}^i = f^i(x, u, a), \quad \bar{u}^{\alpha} = \phi^{\alpha}(x, u, a), \quad i = 1, ..., m, \tag{2.10}$$

that leaves (2.8) form-invariant in new variables $\bar{u}$ and $\bar{x}$, i.e.,

$$\bar{E}^{\sigma}(\bar{x}, \bar{u}, \bar{u_{(1)}}, ..., \bar{u_{(k)}}) = 0, \qquad \sigma = 1, ..., s, \tag{2.11}$$

where (2.8) is satisfied. Here f^i, ϕ^{α} are differentiable functions and $a \in \mathbb{R}$ is a parameter that continuously takes values in a neighborhood $\mathcal{D} \subset \mathbb{R}$ of $a = 0$.

Definition 2.13. *[67] A set G of transformation (2.10), namely,*

$$T_a : \bar{x}^i = f^i(x, u, a), \bar{u}^{\alpha}(x, u, a), \;\; i = 1, ..., n; \alpha = 1, 2, ..., m, \tag{2.12}$$

is a continuous one-parameter (local) Lie group of transformation in $\mathbb{R}^{n+m}$ provided the group properties of closure, identity, and inverses are satisfied, namely,

i. Closure: If $T_a, T_b \in G$, where $a, b \in \mathcal{D}' \subset \mathcal{D}$, then

$$T_a T_b = T_c \in G, \text{where } c = \varphi(a, b) \in \mathcal{D}.$$

ii. Identity: There exists $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a,$$

for any $a \in \mathcal{D}' \subset \mathcal{D}$. T_0 is known as the identity of the group.

iii. Inverse: There exists $T_a^{-1} = T_{a^{-1}} \in G$, where $a^{-1} \in \mathcal{D}$ such that

$$T_a^{-1} T_a = T_a T_{a^{-1}} = T_0,$$

for any $T_a \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$. T_a^{-1} is known as the inverse of the group.

The associativity property follows from (i). Also, if the identity transformation occurs at $a = a_0 \neq 0$, that is, T_{a_0}, then a change of the parameter $a = \bar{a} + a_0$ gives T_0 an in (ii).

We will now refer to a continuous one-parameter(local) group as "a one-parameter group".

Definition 2.14. *A group parameter a is canonical if the group composition law is additive, i.e., $\varphi(a, b) = a + b$.*

If a is canonical, then

$$T_b T_a = T_{a+b} = T_{b+a} = T_a T_b.$$

Theorem 2.1. *[67] For any composition law $\varphi(a, b)$, there exists a (unique) canonical parameter $\tilde{a}$ defined by*

$$\tilde{a} = \int_0^a \frac{da'}{\mathcal{X}},$$

where

$$\mathcal{X}(a') = \left.\frac{\partial \varphi(a', b)}{\partial b}\right|_{b=0}.$$

Proof. See [67]. □

Corrolary 2.4.1. *[60] Any one-parameter group of transformations is abelian, i.e.,*

$$T_a T_b = T_b T_a.$$

Proof. See [5]. □

2.4.2 Infinitesimal Generator and Lie's Equation

According to Lie's theory, constructing the symmetry group G is identical to determining its infinitesimal transformations:

$$\bar{t} \approx t + a\tau(t, x, u), \quad \bar{x} \approx x + a\xi(t, x, u), \quad \bar{u} \approx u + \eta(t, x, u). \tag{2.13}$$

These transformations are obtained by expanding (2.10) in powers of a around $a = 0$ (in a neighborhood where $a = 0$ represents the identity) and setting:

$$\left.\frac{\partial f(t,x,u,a)}{\partial a}\right|_{a=0} = \tau(t,x,u), \quad \left.\frac{\partial g(t,x,u,a)}{\partial a}\right|_{a=0} = \xi(t,x,u), \quad \left.\frac{\partial h(t,x,u,a)}{\partial a}\right|_{a=0} = \eta(t,x,u).$$

The components τ, ξ, and η are referred to as the infinitesimals of (2.12). The vector (τ, ξ, η) represents the tangent vector at the point (t, x, u) to the surface curve described by the transformed points (t, x, u). Hence, it is called the tangent vector field of the group G.

One can now introduce the symbol X (after Lie) of the infinitesimal transformations (2.13) by writing:

$$\bar{t} \approx (1 + X)t, \quad \bar{x} \approx (1 + X)x, \quad \bar{u} \approx (1 + X)u, \tag{2.14}$$

where

$$\begin{aligned} X &= (\tau(t,x,u), \xi(t,x,u), \eta(t,x,u)) \cdot \nabla, \\ &= \tau(t,x,u)\frac{\partial}{\partial t} + \xi(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \end{aligned} \tag{2.15}$$

and the operator ∇ is the gradient vector operator $\nabla = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)$. The operator (2.15) is known as the infinitesimal operator or generator of the group G.

Remark 2.4.1. $X((t,x,u)) = (\tau(t,x,u), \xi(t,x,u), \eta(t,x,u))$.

Theorem 2.2. *[5] Given an infinitesimal transformation (2.13), or an infinitesimal operator(2.15), the transformations (2.11) of the corresponding group G are determined by solving the Lie equations*

$$\frac{d\bar{t}}{da} = \tau(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{x}}{da} = \xi(\bar{t}, \bar{x}, \bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t}, \bar{x}, \bar{u}), \tag{2.16}$$

subject to the initial conditions

$$f(t,x,u,a)|_{a=0} = t, \quad g(t,x,u,a)|_{a=0} = x, \quad h(t,x,u,a)|_{a=0} = u.$$

A one-parameter Lie group of transformation is equivalent to its infinitesimal operator, this allows for

the representation of the solution of Lie equations in terms of the Taylor series (exponential map),

$$\bar{t} = \exp(aX)t, \quad \bar{x} = \exp(aX)x, \quad \bar{u} = \exp(aX)u, \tag{2.17}$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2!} + \cdots = \sum_{j=0}^{\inf} \frac{a^j}{j!} X^j, \tag{2.18}$$

is known as the *Lie series operator.*

2.4.3 Invariants

Definition 2.15. *A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point if it remains unchanged by every transformation of a group G, i.e.,*

$$(\bar{t}, \bar{x}, \bar{u}) = (f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = (t, x, u), \quad \forall a \in \mathcal{D}' \subset \mathcal{D}. \tag{2.19}$$

Theorem 2.3. *[67] A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point of a group G with operator*

$$X = \tau(x, t, u)\frac{\partial}{\partial t} + \xi(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial u}, \tag{2.20}$$

if and only if

$$\tau(t, x, u) = \xi(t, x, u) = \eta(t, x, u). \tag{2.21}$$

Proof. See [67]. □

Definition 2.16. *A function $F(t, x, u)$ is called an invariant of a group G if and only if*

$$F(\bar{t}, \bar{x}, \bar{u}) = F(f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = F(t, x, u), \tag{2.22}$$

for $t, x, u, a \in \mathcal{D}' \subset \mathcal{D}$.

Theorem 2.4. *A function $F(t, x, u)$ is an invariant of a group G with generator X if and only if*

$X(F) = 0$, *i.e.,*

$$X(F) = \tau \frac{\partial F}{\partial t} = \xi \frac{\partial F}{\partial x} = \eta \frac{\partial F}{\partial u}. \tag{2.23}$$

Condition (2.23) is known as the infinitesimal criterion of invariance. From (2.23), the one-parameter group has 2 functionally independent invariants taken from the left-hand sides of 2 first integrals $I_1(t, x, u) = C_1$ and $I_2(t, x, u) = C_2$ of the characteristic equations for linear PDE (2.23). Any other invariant is a function of $I_1(t, x, u) = C_1$ and $I_2(t, x, u) = C_2$.

2.4.4 Prolongation Formulas

Consider a k-th-order PDE

$$E(t, x, u, u_{(1)}, ..., u_{(k)}) = 0, \tag{2.24}$$

where t and x are two independent variables and u is the dependent variable. Let

$$X = \tau(x, t, u)\frac{\partial}{\partial t} + \xi(x, t, u)\frac{\partial}{\partial x} + \eta(x, t, u)\frac{\partial}{\partial u} \tag{2.25}$$

be the infinitesimal generator of the one-parameter group G of transformation(2.11).

Definition 2.17. *The extended infinitesimal generator $X^{[k]}$ of the prolonged group $G^{[k]}$ on the space $(t, x, u, u_{(1)}, ..., u_{(k)})$ is called the kth prolongation of X, and denoted by*

$$\begin{aligned} X^{[k]} = \tau(t, x, u)\frac{\partial}{\partial t} + \xi(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} + \zeta_i(t, x, u, u_{(1)})\frac{\partial}{\partial u_{(1)}} + \\ \cdots + \zeta_{i_1 \ldots i_k}(t, x, u, u_{(1)}, \ldots, u_{(k)})\frac{\partial}{\partial u_{(i_1 \ldots i_k)}}. \end{aligned} \tag{2.26}$$

The coefficients ζ_s are determined recursively by the prolongation formulae:

$$\begin{aligned}
\zeta_i &= D_i(\eta) - u_j D_i(\rho^j) \\
&= D_i(\eta) - u_t D_i(\tau) - u_x D_i(\xi), \\
\zeta_{ij} &= D_j(\zeta_i) - u_{il} D_j(\rho^l) \\
&= D_i(\zeta_i) - u_{it} D_j(\tau) - u_{ix} D_j(\xi) \\
&\vdots \\
\zeta_{i_1 \dots i_k} &= D_{i_k}(\zeta_{i_1 \dots i_{k-1}}) - u_{i_1 \dots i_{k-1} l} D_{i_k}(\rho^l) \\
&= D_{i_k}(\zeta_{i_1 \dots i_{k-1}}) - u_{i_1 \dots i_{k-1} t} D_{i_k}(\tau) - u_{i_1 \dots i_{k-1} x} D_{i_k}(\xi),
\end{aligned} \tag{2.27}$$

where $i, j, i_1, \cdots, i_k = t, x$ and the total derivative operator D_i is given by

$$D_i = \frac{\partial}{\partial i} + u_j \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots .$$

For example, if we consider a second-order partial differential equation (2.8), the second prolongation of the prolonged group $G^{[2]}$ in a space $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ is given by

$$X^{[2]} = X + \zeta_t \partial_{u_t} + \zeta_x \partial_{u_x} + \zeta_{tt} \partial_{u_{tt}} + \zeta_{tx} \partial_{u_{tx}} + \zeta_{xx} \partial_{u_{xx}}. \tag{2.28}$$

The coefficients $\zeta_t, \zeta_x, \zeta_{tx}$ and ζ_{xx} are given by the formulae

$$\begin{aligned}
\zeta_t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi), \\
\zeta_x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi), \\
\zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\tau) - u_{tx} D_t(\xi), \\
\zeta_{tx} = \zeta_{xt} &= D_x(\zeta_t) - u_{tt} D_x(\tau) - u_{tx} D_x(\xi), \\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi),
\end{aligned}$$

where D_t and D_x are total derivatives with respect to t and x respectively given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots,$$
$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots.$$

Using the definitions of the total derivatives D_t and D_x yields

$$\zeta_t = \eta_t + u_t (\eta_u - \tau_t) - u_t^2 \tau_u - u_x \xi_t - u_x u_t \xi_u \tag{2.29}$$

$$\zeta_x = \eta_x + u_x (\eta_u - \xi_x) - u_x^2 \xi_u - u_t \tau_x - u_x u_t \tau_u, \tag{2.30}$$

$$\begin{aligned} \zeta_{tt} = {} & \eta_{tt} + 2u_t \eta_{tu} + u_{tt} \eta_u + u_t^2 \eta_{uu} - 2u_{xt} \xi_t - u_x \xi_{tt} - 2u_x u_t \xi_{tu}^2 - \xi_u (u_x u_{tt} + 2u_t u_{xt}) \\ & - u_x u_t^2 \xi_{uu} - 2u_{tt} \tau_t - u_t \tau_{tt} - 2u_t^2 \tau_{tu} - 3u_t u_{tt} \tau_u - u_t^3 \tau_{uu} \end{aligned} \tag{2.31}$$

$$\begin{aligned} \zeta_{tx} = {} & \eta_{tx} + u_t \eta_{xu} + u_x \eta_{tu} + u_{xt} \eta_u + u_t u_x \eta_{uu} - u_{tt} \tau_x - u_{tx} \tau_t - \tau_u (u_x u_{tt} + 2u_t u_{tx}) \\ & - u_t (\tau_{xt} + u_t \tau_{xu} + u_x \tau_{tu} + u_t u_x \tau_{uu}) - u_{tx} \xi_x - u_{xx} \xi_t - \xi_u (u_t u_{xx} + 2u_x u_{xt}) \\ & - u_x (\xi_{tx} + u_t \xi_{xu} + u_x \xi_{tu} + u_t u_x \xi_{uu}), \end{aligned} \tag{2.32}$$

$$\begin{aligned} \zeta_{xx} = {} & \eta_{xx} + 2u_x \eta_{xu} + u_{xx} \eta_u + u_x^2 \eta_{uu} - 2u_{xx} \xi_x - u_x \xi_{xx} - 2u_x^2 \xi_{xu} - \tau_u (u_t u_{xx} + 2u_x u_{xt}) \\ & - u_x^3 \xi_{uu} - 2u_{xt} \tau_x - u_t \tau_{xx} - 2u_x u_t \tau_{xu} - u_x^2 u_t \tau_{uu} - 3u_x u_{xx} \xi_u \end{aligned} \tag{2.33}$$

2.4.5 Determining Equations for Lie Point Symmetries

In this section, we introduce Lie's algorithm for calculating Lie point symmetries of differential equations.

Definition 2.18. *An invertible transformation acting on the space (t, x, u) of E is a point symmetry of E provided it transform any solution of the equation into another solution of the same equation.*

Theorem 2.5. *[67] Let G be a group of infinitesimal transformations (2.12) admitted by E. Then G consists of symmetries of E if and only if*

$$X^{[2]}(E) = 0, \tag{2.34}$$

whenever (2.8) *is satisfied for every group generator* X *of* G*. Conversely, the operator* X *is a point symmetry of* (2.8) *if*

$$X^{[2]}(E) = 0, \quad \textit{whenever } E = 0. \tag{2.35}$$

The symmetry condition (2.35), also called the invariance criterion can be written compactly as

$$X^{[2]}(E)|_{E=0} = 0, \tag{2.36}$$

where $|_{E=0}$ means evaluated at the equation $E = 0$.

Definition 2.19. *The equation* (2.36) *is called the determining equation, it is an over-determined system of linear homogeneous PDEs for the unknowns* τ, ξ, η *(infinitesimals) of the symmetry generator* X.

Theorem 2.6. *[67] The solution of the determining system forms a vector space, that is, any finite linear combination of the symmetries is again a symmetry.*

Theorem 2.5 summarizes Lie's algorithm. Lie's algorithm for calculating Lie point symmetries of differential equations is explained in several textbooks, among them [5, 32, 67, 6]. We will adopt Lie's algorithm as explained in [60].

Lie's algorithm[60]

1. Write E such that all the terms are on the left-hand side.

2. Write the symmetry generator

$$X = \tau(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u}.$$

3. Prolong the symmetry generator X to the order that is the same as the order of E, i.e.,

$$X^{[k]} = X + \zeta_i\left(t, x, u, u_{(1)}\right)\frac{\partial}{\partial u_{(1)}} + \cdots + \zeta_{i_1\cdots i_k}\left(t, x, u, u_{(1)}, \cdots, u_{(k)}\right)\frac{\partial}{\partial u_{(i_1\cdots i_k)}},$$

   where the variables ζ_i are given by (1.20).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated at $E = 0$, which gives the symmetry condition

$$X^{[k]}(E)\big|_{E=0} = 0.$$

5. Substitute the ζ_i upon expansion of the symmetry condition and replace the derivatives which will be eliminated.

6. Separate the expanded expression concerning the derivatives of the dependent variables and their powers, resulting in an over-determined system of linear homogeneous PDEs in terms of infinitesimals τ, ξ, and η.

7. Solve the over-determined system for the infinitesimals τ, ξ, and η.

8. Construct the one-parameter group using Theorem 2.2.

2.4.6 Lie Algebra

Definition 2.20. *A Lie algebra is a vector space L over a field $\mathbb{F}$ with a binary operation $[-,-] : L \times L \to L$ called the Lie bracket (also known as the commutator), such that the following axioms are satisfied:*

1. ***Bilinearity**: If $X_1, X_2, X_3 \in L$ and $a, b \in \mathbb{F}$, then*

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3].$$

2. ***Skew-Symmetry**: If $X_1 \in L$, then*

$$[X_1, X_1] = 0$$

and this implies that, for all $X_1, X_2 \in L$,

$$[X_1, X_2] = -[X_2, X_1].$$

3. ***Jacobi Identity***: *If* $X_1, X_2, X_3 \in L$, *then*

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

Definition 2.21. *A Lie bracket (commutator)* $[-,-]$ *on the set of vector fields* $\mathcal{V}$ *is defined by*

$$[X_1, X_2] = X_1(X_2) - X_2(X_1),$$

where X_1 *and* X_2 *are operators defined respectively by*

$$X_1 = \tau_1(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \eta_1(t,x,u)\frac{\partial}{\partial u}$$

and

$$X_2 = \tau_2(t,x,u)\frac{\partial}{\partial t} + \xi_2(t,x,u)\frac{\partial}{\partial x} + \eta_2(t,x,u)\frac{\partial}{\partial u}.$$

Theorem 2.7. *[5, 67] The set of all symmetries forms a Lie algebra called a symmetry Lie algebra.*

Proof. See [67]. □

Definition 2.22. *The dimension of a Lie algebra* L *is the dimension of the finite vector space* L*. A finite-dimensional Lie algebra of dimension* r *is denoted by* L_r*.*

Definition 2.23. *Let* L_r *be a Lie algebra over the field* $\mathbb{F}$*. Then a linear subspace* S *of* L $(S \subseteq L)$ *is a subalgebra of* L *if it is closed under the Lie bracket of* L*; that is,* $[X_1, X_2] \in S$ *if* $X_1, X_2 \in S$*.*

Definition 2.24. *Let* I *be a linear subspace of a Lie algebra* L*. Then* I *is an ideal of* L *if* $[X_1, X_2] \in I$ *whenever* $X_1 \in I$ *and* $X_2 \in L$*.*

Definition 2.25. *A Lie algebra* L *is called abelian if the Lie bracket vanishes for all elements in* L*; that is,* $[X_1, X_2] = 0$ *for all* $X_1, X_2 \in L$*.*

2.4.7 Invariant Solution

Definition 2.26. *A solution $u = F(t, x)$ of E is invariant under the one-parameter group of transformations if*

$$\bar{u} = F(\bar{t}, \bar{x}). \tag{2.37}$$

Definition 2.27. *A function $u = F(t, x)$ is said to be an invariant solution under the one-parameter group of transformations (1.3) of E if and only if*

$$X(u - F(t, x))|_{u=F(t,x)} = 0. \tag{2.38}$$

3

Model Framework

In this chapter, we derive the parabolic system that includes the liquidity risk developed by Ludkovski and Shen [54]. The model includes liquidity risk, reflecting market incompleteness. By incorporating an investor's risk preferences through an exponential utility function, we employ a parabolic system to capture the dynamics of option prices under different market conditions.

3.1 The Market Model

We adopt a regime-switching Markov model to capture the dynamic of market liquidity, which involves two distinct states: the liquid state (0) and the illiquid state (1) [74]. This aspect of our model connects with the growing literature on Markov-modulated financial markets driven by the recognition that the economic environment is not stable. In the liquid state, the market operates smoothly, without any hindrances, enabling investors to trade stocks in any desired quantity without influencing stock prices.

Within the framework of the utility maximization, derivative prices are characterized by the Hamilton-Jacobi-Bellman (HJB) equations, which typically take the form of nonlinear parabolic partial differ-

ential equations. In the context of a regime-switching Markov model, the HJB equations result in a coupled system of nonlinear parabolic partial differential equations [53, 54, 74].

3.1.1 Exponential Indifference Pricing

So in our case where the investor's preferences are modeled using the exponential utility function:

$$U(x) = -e^{-\gamma x},$$

where γ is the risk aversion parameter. Exponential utility offers several benefits, including generating wealth-independent prices and being linked through dual representation to the *minimal entropy martingale measure*[1].

The investor aims to maximize the utility of terminal wealth and option payoff at the future horizon $T < \infty$, aligning with the expiration date of all market securities [74]. A starting point for the approach of utility-based pricing are the value functions $\hat{U}_i(t, X, S)$ for the optimal investment problem in the presence of a contingent claim,

$$\hat{U}_i(t, X, S) = \sup_{(\pi_t)\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}_{t,X,S,i}\left[-e^{-\gamma(X_T+h(S_T))}\right], \quad i = 0, 1, \tag{3.1}$$

where $\mathbb{P}$ is the real-world probability, X is the total wealth, π_t is the stock holdings as proportion of total wealth, S_t is the stock price at time $t \in [0, T]$, and $h(S_T)$ in the payoff at terminal time T. Standard stochastic control methods imply the following:

Proposition 3.1. *[54] The optimal value function $\hat{U}_i(t, X, S)$ with $(t, X, S) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+$ for a holder of a bounded contingent claim paying off $h(S_t)$ at terminal time T are given by*

$$\hat{U}_i(t, X, S) = -e^{\gamma X} e^{-\gamma R^i(t,S)}, \tag{3.2}$$

[1] the risk neutral measure that minimises the entropy difference between the objective probability measure and the risk neutral measure.

where $R^i(t,S)$ are the unique viscosity solutions of the coupled semi-linear system,

$$\begin{aligned} R_t^0 + \frac{1}{2}\sigma_0^2 S^2 R_{SS}^0 - \frac{\nu_{01}}{\gamma} e^{-\gamma(R^1-R^0)} + \frac{(d_0+\nu_{01})}{\gamma} &= 0, \\ R_t^1 - \frac{\nu_{10}}{\gamma} e^{-\gamma(R^0-R^1)} + \frac{\nu_{10}}{\gamma} &= 0, \end{aligned} \tag{3.3}$$

where $R^i(t,S), \quad i=0,1$ define the price of the options in two states, σ is the volatility of the stock, ν_{01}, ν_{10} are the transition from liquid state (0) to the illiquid state (1) and vice-versa, respectively, γ is the risk aversion parameter, and $d_0 = \frac{\mu^2}{2\sigma^2}$ where μ is the drift of the underlying stock. The terminal conditions are given as

$$R^i(S,T) = h(S), \quad i=0,1. \tag{3.4}$$

The system (3.3) and the conditions (3.4) are solved for the stock price $S \geq 0$ of the underlying asset with $t \in [0,T]$.

Proof. See [54]. □

3.1.2 Indifference Prices

Indifference pricing involves comparing the optimal investment strategies with and without options to determine the price at which an investor is indifferent between the two scenarios [9]. Let $\hat{V}_i$ represent value functions for an investor following an optimal trading strategy but without investment in options. Using $\hat{U}_i$ and $\hat{V}_i$, the buyer's indifference prices p (initial state 0) and q (initial state 1) are defined as:

$$\hat{U}^0(t, X-p, S) = \hat{V}^0(t,X), \quad \hat{U}^1(t, X-q, S) = \hat{V}^1(t,X). \tag{3.5}$$

The value functions $\hat{V}^i, i=0,1$ are derived in [54]. The equation (3.5) imply that the utility of wealth minus the indifference price with options equals the utility of wealth without options. Using the value functions $\hat{U}^i(t,X), \hat{V}^i(t,X)$ and (3.5) we can write the relationship between indifference prices p, q

and R^0, R^1 as follows:

$$p = R^0 + \gamma^{-1} \log Y_0(t), \quad q = R^1 + \gamma^{-1} \log Y_1(t), \tag{3.6}$$

expressions for $Y_i(t)$ in Ludkovski and Shen [54] *Definition 3.2* are given as

$$Y_0(t) = c_1 e^{-\lambda_1 (T-t)} + c_2 e^{-\lambda_2 (T-t)}, \tag{3.7}$$

$$Y_1(t) = \frac{1}{\nu_{01}} \left[c_1 (d_0 + \nu_{01} - \lambda_1) e^{\lambda_1 (T-t)} + c_2 (d_0 + \nu_{01} - \lambda_2) e^{\lambda_2 (T-t)} \right], \tag{3.8}$$

where

$$\begin{aligned} d_0 &= \frac{\mu^2}{2\sigma^2}, \\ \lambda_{1,2} &= \frac{d_0 + \nu_{01} + \nu_{10} \pm \sqrt{(d_0 + \nu_{01} + \nu_{10})^2 - 4 d_0 \nu_{10}}}{2}, \\ c_1 &= \frac{\lambda_2 - d_0}{\lambda_2 - \lambda_1}, \\ c_2 &= \frac{\lambda_1 - d_0}{\lambda_1 - \lambda_2}. \end{aligned}$$

Using (3.6), the system (3.3) in terms of indifference prices (p, q) can be rewritten as

$$\begin{aligned} p_t + \frac{1}{2}\sigma_0^2 S^2 p_{SS} - \frac{\nu_{01}}{\gamma} e^{-\gamma(q-p)} + \frac{(d_0 + \nu_{01})}{\gamma} &= 0, \\ q_t - \frac{\nu_{10}}{\gamma} e^{-\gamma(p-q)} + \frac{\nu_{10}}{\gamma} &= 0, \end{aligned} \tag{3.9}$$

with terminal conditions,

$$p(T, S) = q(T, S) = h(S). \tag{3.10}$$

3.1.3 Formulation of the Model

By making the substitutions $S = x$ and transforming the option prices as

$$u = \gamma R_0, \quad v = \gamma R_1,$$

then substituting these transformations into the initial system (3.3), we obtain the following system of partial differential equations (PDEs) that describe the evolution of the option prices under the influence of liquidity shocks:

$$\begin{aligned} u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} + ae^{u-v} - b = 0 \\ v_t + ce^{v-u} - c = 0 \end{aligned} \tag{3.11}$$

where $a = \nu_{01}$, $b = d_0 + \nu_{01}$, and $c = \nu_{10}$. In accordance with (3.9), we take the initial conditions to be

$$u(0, x) = u_0(x) = \gamma h(x), \quad v(0, x) = v_0(x) = \gamma h(x). \tag{3.12}$$

For a call option,

$$h(x) = \max(x - K, 0), \tag{3.13}$$

where K is the strike price.

3.2 Conclusion

We derived a model for pricing European options under liquidity shocks, extending Ludkovski and Shen's framework by modeling liquidity risk with a regime-switching Markov process that captures liquid and illiquid market states, deriving nonlinear parabolic PDEs from the Hamilton-Jacobi-Bellman equations considering the investor's exponential utility and market incompleteness, which enables option pricing through indifference pricing that adjusts for liquidity impacts. .

4

Lie Symmetry Analysis Applied to the Coupled Systems in the Modeling of European Options

In this chapter, we consider the system of partial differential equations–derived in the previous chapter,

$$\begin{aligned} u_t &= \frac{1}{2}\sigma^2 x^2 u_{xx} - ae^{u-v} + b, \\ v_t &= c - ce^{v-u}, \end{aligned} \tag{4.1}$$

with initial conditions; $u(0,x) = u_0(x) = \gamma h(x), \quad v(0,x) = v_0(x) = \gamma h(x)$, where $h(x) = \max(x-K,0)$ and K is the strike price.

We carry out Lie symmetry analysis, which is the process of calculating Lie point symmetries, determining optimal systems, carrying out symmetry reductions, and creating group invariant solutions.

4.1 Lie Point Symmetries

In the space of independent and dependent variables, we examine a one-parameter Lie group of infinitesimal transformations as follows and defined in [6, 32, 60, 67]:

$$\begin{aligned}
\overline{x} &= x + \epsilon\xi(x,t,u,v) + O(\epsilon^2),\\
\overline{t} &= t + \epsilon\tau(x,t,u,v) + O(\epsilon^2),\\
\overline{u} &= u + \epsilon\eta_1(x,t,u,v) + O(\epsilon^2),\\
\overline{v} &= v + \epsilon\eta_2(x,t,u,v) + O(\epsilon^2),
\end{aligned} \tag{4.2}$$

where ξ, τ, η_1 and η_2 are infinitesimal generators to be determined and ϵ represents the group parameter. Lie's algorithm states that the vector field

$$\mathbf{V} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}, \tag{4.3}$$

is a symmetry generator of (4.1) if and only if

$$\begin{aligned}
&\mathbf{V}^{[2]}(u_t - \frac{1}{2}\sigma^2x^2u_{xx} + ae^{u-v} - b = 0)\big|_{u_t=\frac{1}{2}\sigma^2x^2u_{xx}-ae^{u-v}+b} = 0,\\
&\mathbf{V}^{[1]}\left(v_t - c + ce^{v-u} = 0\right)\big|_{v_t=c-ce^{v-u}} = 0,
\end{aligned} \tag{4.4}$$

where $\mathbf{V}^{[2]}$ is the second prolongation of $\mathbf{V}$ to the system given by,

$$\mathbf{V}^{[2]} = \mathbf{V} + \zeta_1^1\partial u_t + \zeta_2^1\partial u_x + \zeta_1^2\partial v_t + \zeta_2^2\partial v_x + \zeta_{11}^1\partial u_{tt} + \zeta_{12}^1\partial u_{xt} + \zeta_{22}^1\partial u_{xx} + \zeta_{11}^2\partial v_{tt} + \zeta_{12}^2\partial u_{tt} + \zeta_{22}^2\partial v_{xx}.$$

The variables ζ_j^i are given by the prolongation formulae:

$$\zeta_1^1 = D_t(\eta_1) - u_tD_t(\tau) - u_xD_t(\xi), \tag{4.4a}$$

$$\zeta_2^1 = D_x(\eta_1) - u_tD_x(\tau) - u_xD_x(\xi), \tag{4.4b}$$

$$\zeta_1^2 = D_t(\eta_2) - v_tD_t(\tau) - v_xD_t(\xi), \tag{4.4c}$$

$$\zeta_2^2 = D_x(\eta_2) - v_tD_x(\tau) - v_xD_x(\xi), \tag{4.4d}$$

$$\zeta^1_{11} = D_t(\zeta^1_1) - u_{tt}D_t(\tau) - u_{xt}D_t(\xi), \tag{4.4e}$$

$$\zeta^1_{12} = D_x(\zeta^1_1) - u_{tt}D_x(\tau) - u_{xt}D_x(\xi), \tag{4.4f}$$

$$\zeta^1_{22} = D_x(\zeta^1_2) - u_{tx}D_x(\tau) - u_{xx}D_x(\xi), \tag{4.4g}$$

$$\zeta^2_{11} = D_t(\zeta^2_1) - v_{tt}D_t(\tau) - v_{xt}D_t(\xi), \tag{4.4h}$$

$$\zeta^2_{12} = D_t(\zeta^2_1) - v_{tt}D_x(\tau) - v_{xt}D_x(\xi), \tag{4.4i}$$

$$\zeta^2_{22} = D_x(\zeta^2_2) - v_{tx}D_x(\tau) - v_{xx}D_x(\xi), \tag{4.4j}$$

where D_t and D_x are the total derivative operators defined as follows:

$$D_t = \partial_t + u_t\partial_u + v_t\partial_v + \cdots, \tag{4.4k}$$

$$D_x = \partial_x + u_x\partial_u + v_x\partial_v + \cdots. \tag{4.4l}$$

Using the above we get the following system of prolonged equations,

$$\begin{aligned}
&ae^{u-v}\eta_1 - ae^{u-v}\eta_2 - u_x\tau_t - u_tu_x\tau_u - u_xv_t\tau_v - u_t\xi_t - u_t^2\xi_u \\
&- u_tv_t\xi_v + \eta_{1t} + u_t\eta_{1u} + v_t\eta_{1v} + x^2\sigma^2u_x\xi_{2u}u_{xt} + x^2\sigma^2v_x\xi_vu_{xt} \\
&+ x^2\sigma^2\xi_xu_{xt} - x\sigma^2\xi_1u_{xx} + 3x^2\sigma^2u_x\tau_uu_{xx} + 2x^2\sigma^2v_x\tau_vu_{xx} \\
&+ x^2\sigma^2\tau_xu_{xx} + \frac{1}{2}x^2\sigma^2u_t\xi_{2u}u_{xx} - \frac{1}{2}x^2\sigma^2\eta_{1u}u_{xx} + \frac{1}{2}x^2\sigma^2u_x\tau_vv_{xx} \\
&+ \frac{1}{2}x^2\sigma^2u_t\xi_vv_{xx} - \frac{1}{2}x^2\sigma^2\eta_{1v}v_{xx} + \frac{1}{2}x^2\sigma^2u_{3x}\tau_{uu} + x^2\sigma^2u_{2x}v_x\tau_{uv} \\
&+ \frac{1}{2}x^2\sigma^2u_xv_{2x}\tau_{vv} + \frac{1}{2}x^2\sigma^2u_{2x}\tau_{xu} + x^2\sigma^2u_xv_x\tau_{xv} + \frac{1}{2}x^2\sigma^2u_x\tau_{xx} \\
&+ \frac{1}{2}x^2\sigma^2u_tu_{2x}\xi_{uu} + x^2\sigma^2u_tu_xv_x\xi_{uv} + \frac{1}{2}x^2\sigma^2u_tv_{2x}\xi_{vv} + x^2\sigma^2u_xv_x\eta_{1uv} \\
&- \frac{1}{2}x^2\sigma^2u_tu_x\xi_{xu} + x^2\sigma^2u_tv_x\xi_{xv} + \frac{1}{2}x^2\sigma^2u_t\xi_{xx} - x^2\sigma^2v_{2x}\eta_{1vv} \\
&- x^2\sigma^2u_x\eta^1_{xu} - x^2\sigma^2v_x\eta_{1xv} = 0, \\
&- ce^{-u+v}\eta_1 + ce^{-u+v}\eta_2 - v_x\tau_t - u_tv_x\tau_u - v_tv_x\tau_v \\
&- v_t\xi_t - u_tv_t\xi_u - v_t^2\xi_v + \eta_{2t} + u_t\eta_{2u} + v_t\eta_{2v} = 0.
\end{aligned}$$

4.2 Determining Equations for Lie Symmetries

Although it takes a while and is tedious, manually finding the determining equations is simple. Due to the algorithmic nature of the Lie method stated in Subsection 2.4.5, programs for symbolic manipulation such as Mathematica, Maple, Maxima, and Reduce have been developed to perform this task. In our work, we will be using the Mathematica 12.1 [79] software package *SYM* [13]. Following Subsection 2.4.5, the determining equations are

$$\xi_u = \xi_v = 0, \tag{4.5}$$

$$\tau_u = \tau_x = \tau_v = 0, \tag{4.5a}$$

$$\xi_{uu} = \xi_{vv} = \xi_{uv} = 0, \tag{4.5b}$$

$$\tau_{uu} = \tau_{vv} = \tau_{uv} = 0, \tag{4.5c}$$

$$ae^u\tau_v - be^v\tau_v + e^v\eta_{1v} = 0, \tag{4.5d}$$

$$ae^u\tau_{vv} - be^v\tau_{vv} + e^v\eta_{1vv} - 2ae^u\tau_v = 0, \tag{4.5e}$$

$$ae^u\tau_{xv} - be^v\tau_{xv} + e^v\eta_{1xv} + ae^u\tau_x = 0, \tag{4.5f}$$

$$-ce^u\tau_u + ce^v\tau_u + e^u\eta_{2u} = 0, \tag{4.5g}$$

$$2\xi_u + x^2\sigma^2\tau_{xu} + 2x\sigma^2\tau_u = 0, \tag{4.5h}$$

$$2\xi_v + x^2\sigma^2\tau_{xv} + 2x\sigma^2\tau_v = 0, \tag{4.5i}$$

$$-2e^v\xi_{xu} + ae^u\tau_{uu} - be^v\tau_{uu} + e^v\eta_{1uu} = 0, \tag{4.5j}$$

$$-e^v\xi_{xv} + ae^u\tau_{uv} - be^v\tau_{uv} + e^v\eta_{1uv} = 0, \tag{4.5k}$$

$$-e^{u+v}\xi_t + ae^{2u}\xi_u - be^{u+v}\xi_u + ce^{2v}\xi_v - ce^{u+v}\xi_v = 0, \tag{4.5l}$$

$$\begin{aligned}-4e^{u+v}\xi - 4e^{u+v}x\xi_x + 2e^{u+v}x\tau_t - 2ae^{2u}x\tau_u + 2be^{u+v}x\tau_u - 2ce^{2v}x\tau_v + \\ 2ce^{u+v}x\tau_v - e^{u+v}x^3\sigma^2\tau_{xx} = 0,\end{aligned} \tag{4.5m}$$

$$\begin{aligned}-2e^{u+v}\xi_t + 2ae^{2u}\xi_u - 2be^{u+v}\xi_u + 2ce^{2v}\xi_v - 2ce^{u+v}\xi_v + e^{u+v}x^2\sigma^2\xi_{xx} - \\ 2ae^{2u}x^2\sigma^2\tau_{xu} + 2be^{u+v}x^2\sigma^2\tau_{xu} - 2e^{u+v}x^2\sigma^2\eta_{1xu} = 0,\end{aligned} \tag{4.5n}$$

$$ce^{u+2v}\eta_1 - ce^{u+2v}\eta_2 + ce^{2u+v}\tau_t - ce^{u+2v}\tau_t - ace^{3u}\tau_u + ace^{2u+v}\tau_u +$$
$$bce^{2u+v}\tau_u - bce^{u+2v}\tau_u + c^2e^{3v}\tau_v + c^2e^{2u+v}\tau_v - 2c^2e^{u+2v}\tau_v - e^{2u+v}\eta_{2t} +$$
$$ae^{3u}\eta_{2u} - be^{2u+v}\eta_{2u} - ce^{2u+v}\eta_{2v} + ce^{u+2v}\eta_{2v} = 0, \tag{4.5o}$$

$$-2ae^{2u+v}\eta_1 + 2ae^{2u+v}\eta_2 - 2ae^{2u+v}\tau_t + 2be^{u+2v}\tau_t + 2a^2e^{3u}\tau_u - 4abe^{2u+v}\tau_u +$$
$$2b^2e^{u+2v}\tau_u - 2bce^{3v}\tau_v - 2ace^{2u+v}\tau_v + 2ace^{u+2v}\tau_v + 2bce^{u+2v}\tau_v - 2e^{u+2v}\eta_{1t} +$$
$$2ae^{2u+v}\eta_{1u} - 2be^{u+2v}\eta_{1u} + 2ce^{3v}\eta_{1v} - 2ce^{u+2v}\eta_{1v} + ae^{2u+v}x^2\sigma^2\tau_{xx} -$$
$$be^{u+2v}x^2\sigma^2\tau_{xx} + e^{u+2v}x^2\sigma^2\eta_{1xx} = 0. \tag{4.5p}$$

We observe from equations (4.5) and (4.5a) that,

$$\xi = \xi(x,t), \tag{4.6}$$
$$\tau = \tau(t). \tag{4.7}$$

Substituting (4.6) and (4.7) in (4.5b) through (4.5p) we get that,

$$e^{u+v}\xi_t = 0, \tag{4.8}$$
$$e^v\eta_{1v} = 0, \tag{4.8a}$$
$$e^v\eta_{1vv} = 0, \tag{4.8b}$$
$$e^u\eta_{2u} = 0, \tag{4.8c}$$
$$e^v\eta_{1uv} = 0, \tag{4.8d}$$
$$e^v\eta_{1uu} = 0, \tag{4.8e}$$
$$e^v\eta_{1xv} = 0, \tag{4.8f}$$
$$e^{u+v}(2\xi + x(\tau_t - \xi_x)) = 0, \tag{4.8g}$$
$$e^{u+v}(-2\xi_t + x^2\sigma^2(\xi_{xx} - 2\eta_{1xu})) = 0, \tag{4.8h}$$
$$e^u(-ce^{2v}\eta_1 + ce^{2v}\eta_2 + ce^{2v}\tau_t - ce^{u+v}\tau_t - ce^{2v}\eta_{2v} +$$
$$ce^{u+v}\eta_{2v} - ae^{2u}\eta_{2u} + be^{u+v}\eta_{2u} + e^{u+v}\eta_{2t}) = 0, \tag{4.8i}$$

$$e^{v}(-2ae^{2u}\eta_1 + 2ae^{2u}\eta_2 - 2ae^{2u}\tau_t + 2be^{u+v}\tau_t + 2ce^{2v}\eta_{1v} - 2ce^{u+v}\eta_{1v} + 2ae^{2u}\eta_{1u} - 2be^{u+v}\eta_{1u} - 2e^{u+v}\eta_{1t} + e^{u+v}x^2\sigma^2\eta_{1xx}) = 0. \tag{4.8j}$$

Solving the above equations, the desired infinitesimals generators on the parabolic system (4.1) are as follows,

$$\xi = xk_1, \tag{4.9}$$

$$\tau = k_2, \tag{4.10}$$

$$\eta_1 = k_4x + k_3, \tag{4.11}$$

$$\eta_2 = k_4x + k_3. \tag{4.12}$$

The symmetry generator is given by

$$\mathbf{V} = k_1x\partial_x + k_2\partial_t + (k_4x + k_3)\partial_u + (k_4x + k_3)\partial_v. \tag{4.13}$$

Consequently, following the Lie symmetry method explained in [67], we obtain the following 4-dimensional classical algebra $\mathcal{L}^4$ of the parabolic system (4.1) with the aid of (4.13):

$$\begin{aligned} v_1 &= x\partial_x, \\ v_2 &= \partial_t, \\ v_3 &= \partial_u + \partial_v, \\ v_4 &= x\partial_u + x\partial_v. \end{aligned} \tag{4.14}$$

4.3 A One-dimensional Optimal System of Subalgebras

In this section, we develop a one dimensional optimal system of the algebra $\mathcal{L}^4$ obtained in Section 4.2 spanned by the parabolic system (4.1). This symmetry Lie algebra is used to perform symmetry reductions and construct invariant solutions. Certain linear combinations of the Lie point symmetries can be used to for symmetry reductions to get invariant solutions.

In some cases, a system may admit numerous Lie point symmetries, leading to a larger number of possible linear combinations. However, certain symmetries are related through transformations within the symmetry group, implying that their corresponding invariant solutions are also related by these transformations. To manage this redundancy, symmetries that are equivalent under such transformations are grouped into the same class. The structured method of finding these equivalence classes is known as constructing an optimal system of subalgebras [73].

For the one-dimensional subalgebras, this classification problem is essentially the same as the process of classifying orbits of the adjoint representation, since each one dimensional subalgebra is determined by a nonzero vector in the algebra. Essentially, this process begins with a general element $\mathbf{V}$ of the Lie algebra, which is then subjected to a series of adjoint transformations with the goal of simplifying it as much as possible [62].

Now, in our case, given a nonzero vector

$$\mathbf{V} = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 \tag{4.15}$$

in algebra spanned by the parabolic system. The vector corresponds to the vector $\mathbf{a} = (a_1, a_2, a_3, a_4)$. Our task is to simplify as many of the coefficients a_i being zero as possible through application of the adjoint maps on $\mathbf{V}$ and find its equivalent one.

In this thesis we use the matrix method given in [32] to determine an optimal system of the one-dimensional subalgebra of $\mathcal{L}^4$. The algorithm is given by the following steps.

1. Construction of the commutator table and determining the structure constants matrix $C(j)$ by the formula

$$(C(j))^k_j = c^k_{ij}, \tag{4.16}$$

   where c^k_{ij} is the structure constants given in the commutator table.

2. Calculating the adjoint matrix $A(j, \epsilon)$ by definition

$$A(j, \epsilon) = exp(\epsilon C(j)) = \sum_{n=0}^{\infty} C(j)^n \frac{\epsilon^n}{n!}. \tag{4.17}$$

3. Simplifying the vector $\mathbf{a} = (a_1, a_2, a_3, a_4)$ as much as possible by applying $A(j, \epsilon)$ on $\mathbf{a}$ and obtain the equivalent one of $\mathbf{V}$.

We proceed by implementing the above steps to construct the optimal system of $\mathcal{L}^4$.

4.3.1 Construction of the Commutator Table

The commutation relations (also known as the Lie Brackets) between these vector fields are obtained by considering a formula provided by [67] is given by,

$$[v_i, v_j] = v_i \cdot v_j - v_j \cdot v_i \tag{4.18}$$

where i represents the entry in row i and j the entry in column j. We compute some commutations below,

Case$[v_1, v_2]$:

$$\begin{aligned}[v_1, v_2] &= v_1(v_2) - v_2(v_1), \\ &= x\partial_x\left(\partial_t\right) - \partial_t\left(x\partial_x\right), \\ &= 0.\end{aligned} \tag{4.19}$$

Case$[v_1, v_3]$:

$$\begin{aligned}[v_1, v_3] &= v_1(v_3) - v_3(v_1), \\ &= x\partial_x\left(\partial_u + \partial_v\right) - \left(\partial_u + \partial_v\right)\left(x\partial_x\right), \\ &= 0.\end{aligned} \tag{4.20}$$

Case$[v_1, v_4]$:

$$\begin{aligned}
[v_1, v_4] &= v_1(v_4) - v_4(v_1), \\
&= x\partial_x \left(x\partial_u + x\partial_v\right) - \left(x\partial_u + x\partial_v\right)\left(x\partial_x\right), \\
&= x\partial_u + x\partial_v, \\
&= v_4. \qquad (4.21)
\end{aligned}$$

The rest of the commutations of (4.1) upon calculation are shown in Table 4.1 below,

Table 4.1: Commutator table of the parabolic system

$[v_i, v_j]$	v_1	v_2	v_3	v_4
v_1	0	0	0	v_4
v_2	0	0	0	0
v_3	0	0	0	0
v_4	$-v_4$	0	0	0

We observe that Table 4.1 is skew-symmetric with zero diagonal elements and is the direct sum of the Lie algebras.

Commutator of Subalgebras

We consider the case v_2 and v_3, and with the help of the computations of commutators above, we are to construct the Table 4.2 below.

Table 4.2: Commutator table of the Subalgebra $\{v_2, v_3\}$

$[v_i, v_j]$	v_2	v_3
v_2	0	0
v_3	0	0

We do same thing the algebras v_1 and v_4, and construct Table 4.3 below.

The optimal set of the original Lie algebra will consist of the direct sum of the optimal sets of the two subalgebras, since the first subalgebra is abelian so the elements v_2 and v_3 belong to its optimal set, and they will be added later to each of the found subalgebras as a direct sum with arbitrary constants.

Table 4.3: Commutator table of the Subalgebra $\{v_1, v_4\}$

$[v_i, v_j]$	v_1	v_4
v_1	0	v_4
v_4	$-v_4$	0

As the commutator Table 4.3 showing, we have the structure constants matrices for the $\mathcal{L}^4$ as follows

$$C(1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C(2) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

4.3.2 Calculation of the Adjoint Matrix $A(j, \epsilon)$

Now we exponentiate the matrices $\epsilon C(j)$ to obtain $A(j, \epsilon)$ which are as follows

$$A(1, \epsilon) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \quad \text{and} \quad A(2, \epsilon) = \begin{pmatrix} 1 & e^{-\epsilon} \\ 0 & 1 \end{pmatrix}.$$

Before simplifying $\mathbf{a}$, first check for the existence of invariants. The row vectors $aC(j)$ are

$$aC(1) = (0, a_4), \tag{4.22}$$

$$aC(2) = (0, -a_1), \tag{4.23}$$

so any invariants satisfy

$$\begin{bmatrix} 0 & a_4 \\ 0 & -a_1 \end{bmatrix} \begin{bmatrix} I_1(a) \\ I_2(a) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Here $\rho = 2$, and we have a 2×2 matrix, thus there is only one invariant:

$$\begin{aligned} a_4 \frac{\partial I}{\partial a_4} &= 0, \\ a_1 \frac{\partial I}{\partial a_4} &= 0. \end{aligned} \tag{4.24}$$

Using the method of characteristics we find that $I = I(a_1)$. Here the basic invariant of the parabolic

system (4.1) is only one, i.e. a_1.

The vector $\mathbf{a}$ is transformed by the matrices $A(j,\epsilon)$ as follows:

$$aA(1,\epsilon) = (a_1, e^{-\epsilon}a_4), \tag{4.25}$$

$$aA(2,\epsilon) = (a_1, \epsilon a_1 + a_4). \tag{4.26}$$

In matrix form we have

$$\begin{pmatrix} a_1 & e^{-\epsilon}a_4 \\ a_1 & \epsilon a_1 + a_4 \end{pmatrix}. \tag{4.27}$$

Since a_1 is the only invariant, we separate the cases when $a_1 = 1$ and $a_1 = 0$.

When $a_1 = 1$, (4.27) becomes

$$\begin{pmatrix} 1 & e^{-\epsilon}a_4 \\ 1 & \epsilon + a_4 \end{pmatrix}. \tag{4.28}$$

So, we can always find ϵ such that $a_4 = 0$, so the first element of the optimal system is v_1. When $a_1 = 0$, (4.27) results

$$\begin{pmatrix} 0 & e^{-\epsilon}a_4 \\ 0 & a_4 \end{pmatrix}. \tag{4.29}$$

From this orbit (4.29) we see that the second element of the optimal set is v_4. Now, considering these two elements and the abelian elements, the comprehensive optimal system of symmetry algebra (4.14) is described by the set

$$\{v_1 + \alpha v_2 + \beta v_3, v_4 + \alpha v_2 + \beta v_3\}, \tag{4.30}$$

for some arbitrary constants α, β.

4.4 Exact Invariant Solutions

In this section, various closed-form invariant solutions for the corresponding symmetry subalgebras are constructed by solving Lagrange's characteristic equation [67]

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta_1} = \frac{dv}{\eta_2}. \tag{4.31}$$

If we choose $\alpha = 1$ and $\beta = 0$ in the second linear combination of (4.30), we have a subalgebra $\mathfrak{V}_1^1 := v_2 + v_4$. When we choose $\alpha = 0$ and $\beta = 0$, obtain a subalgebra $\mathfrak{V}_1^2 := v_4$ which gives us a solution that does not satisfy the invariant surface condition. A similar solution applies when choosing $\alpha = 0$ and $\beta = 1$. Lastly, when we consider $\alpha = 1$ and $\beta = 1$, in the second linear combination, we have a subalgebra $\mathfrak{V}_1^3 := v_2 + v_3 + v_4$. Following a similar approach, we do the same thing with the first linear combination of (4.30) to get the following subalgebras,

- $\mathfrak{V}_2^1 := v_1$,
- $\mathfrak{V}_2^2 := v_1 + v_3$,
- $\mathfrak{V}_2^3 := v_1 + v_2$,
- $\mathfrak{V}_2^4 := v_1 + v_2 + v_3$.

4.4.1 *Subalgebra* $\mathfrak{V}_1^1 := v_2 + v_4 = \partial_t + (x\partial_u + x\partial_v)$.

The Langrange's system (4.31) becomes

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{x} = \frac{dv}{x},$$

from the first ratio we get that

$$x = c_1. \tag{4.32}$$

Checking of invariant c_1

Checking for the invariant c_1 we have

$$\begin{aligned}\mathfrak{V}_1(c_1) &= \partial_t(c_1) + (x\partial_u(c_1) + x\partial_v(c_1)),\\ &= \partial_t(x) + (x\partial_u(x) + x\partial_v(x)),\\ &= 0.\end{aligned}$$

This operator satisfies the invariant condition. From the second and third ratios, we deduce that

$$c_2 = u - tx. \tag{4.33}$$

Checking of invariant c_2

Checking for the invariant c_2, we have

$$\begin{aligned}\mathfrak{V}_1(c_2) &= \partial_t(c_2) + (x\partial_u(c_2) + x\partial_v(c_2)),\\ &= -1 + 1,\\ &= 0.\end{aligned}$$

This operator also satisfies the invariant condition. Considering the second and fourth ratios yield,

$$c_3 = v - tx. \tag{4.34}$$

Checking of invariant c_3

Checking for the invariant c_3, we have

$$\begin{aligned}\mathfrak{V}_1(c_3) &= \partial_t(c_3) + (x\partial_u(c_3) + x\partial_v(c_3)),\\ &= -1 + 1,\\ &= 0.\end{aligned}$$

This operator also satisfies the invariant condition. Taking into account of (4.32) and (4.33) we notice the following

$$\begin{aligned}c_2 &= \mathcal{F}(c_1),\\ u - tx &= \mathcal{F}(x),\\ u(x,t) &= \mathcal{F}(x) + tx.\end{aligned} \tag{4.35}$$

And combining equations (4.32) and (4.34), we find:

$$\begin{aligned}c_3 &= \mathcal{G}(c_1),\\ v - tx &= \mathcal{G}(x),\\ v(x,t) &= \mathcal{G}(x) + tx,\end{aligned} \tag{4.36}$$

where $\mathcal{F}$ and $\mathcal{G}$ are arbitrary functions. Substituting (4.35) and (4.36) in (4.1) and simplifying we obtain:

$$x = \frac{1}{2}\sigma^2 x^2 \mathcal{F}''(x) - ae^{\mathcal{F}(x)-\mathcal{G}(x)} + b, \tag{4.37}$$

$$x = c - ce^{-(\mathcal{F}(x)-\mathcal{G}(x))}. \tag{4.38}$$

Making $\mathcal{F}(x) - \mathcal{G}(x)$ the subject in (4.38):

$$
\begin{aligned}
ce^{-(\mathcal{F}(x)-\mathcal{G}(x))} &= c-x,\\
e^{-(\mathcal{F}(x)-\mathcal{G}(x))} &= \frac{c-x}{c},\\
\ln(e^{-(\mathcal{F}(x)-\mathcal{G}(x))}) &= \ln\left(\frac{c-x}{c}\right),\\
\mathcal{F}(x)-\mathcal{G}(x) &= \ln\left(\frac{c}{c-x}\right).
\end{aligned}
\tag{4.39}
$$

Substituting (4.39) in (4.37) we obtain:

$$
x+\frac{ac}{c-x} = \frac{1}{2}\sigma^2x^2\mathcal{F}''(x)+b \tag{4.40}
$$

Upon solving equation (4.40) we get:

$$
\mathcal{F}(x) = \frac{2ax+c\sigma^2k_1+cx(\sigma^2k_2-2)+2c(b+x)\ln(x)-2a(x+c)\ln(x)}{c\sigma^2}, \tag{4.41}
$$

where k_1 and k_2 are constants. Now (4.35) become:

$$
u(x,t) = \frac{2ax+c\sigma^2k_1+cx(\sigma^2k_2-2)+2c(b+x)\ln(x)-2a(x+c)\ln(x)}{c\sigma^2}+tx. \tag{4.42}
$$

Substituting (4.41) in (4.39) and solving for $\mathcal{G}$ we obtain:

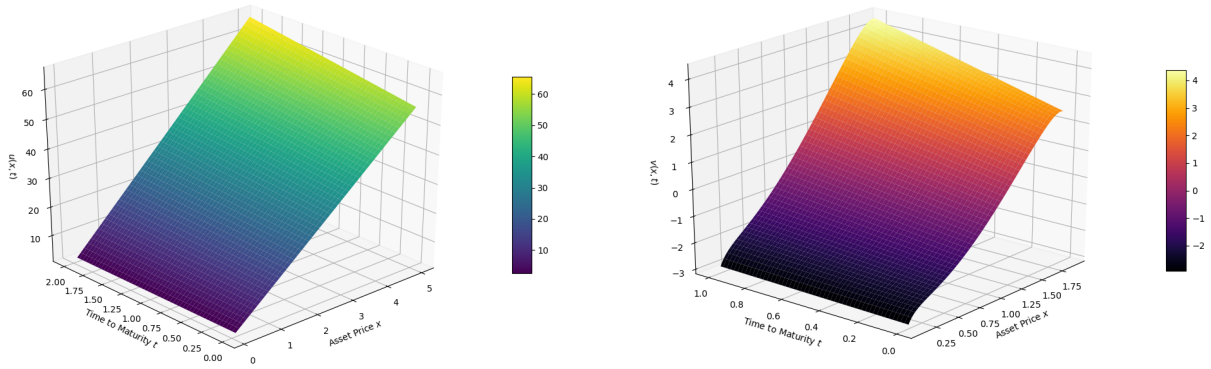
$$
\mathcal{G}(x) = \frac{2ax+c\sigma^2k_1+cx(\sigma^2k_2-2)+2c(b+x)\ln(x)-2a(x+c)\ln(x)}{c\sigma^2}-\ln\left(\frac{c}{c-x}\right) \tag{4.43}
$$

Substituting (4.43) in (4.36), we get the solution to (4.36) as

$$
v(x,t) = \frac{2ax+c\sigma^2k_1+cx(\sigma^2k_2-2)+2c(b+x)\ln(x)-2a(x+c)\ln(x)}{c\sigma^2}-\ln\left(\frac{c}{c-x}\right)+tx. \tag{4.44}
$$

Thus, (4.42) and (4.44) are the invariant solution of (4.1) under the subalgebra $\mathfrak{V}_1$.

Figure 4.1: Comparison of option values: (a) shows the 3D option value solution to (4.42), and (b) presents the 3D option value as a function of stock price and time to (4.44).



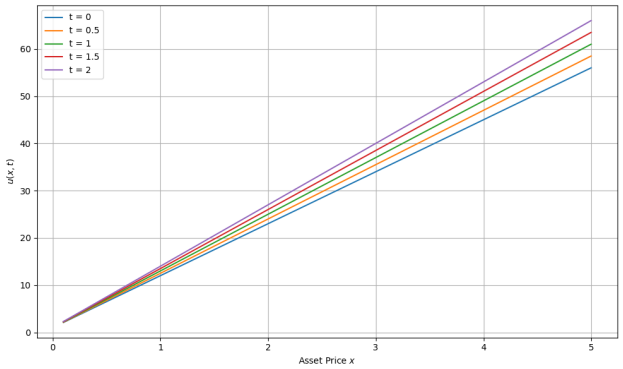
(a) Option Value $u(x,t)$ over Stock Price and Time

(b) Option Value $v(x,t)$ over Stock Price and Time

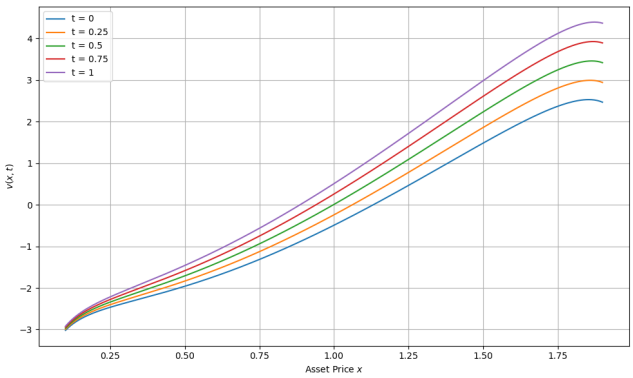
In Figure 4.1a, the parameters were set as $a = 1$, $b = 1$, $c = 1$, $\sigma = 1$, $k_1 = 1$, and $k_2 = 11$. For Figure 4.1b, the parameters were adjusted to $a = 0.8$, $b = 1$, $c = 2$ (where c is chosen to be greater than the maximum asset price x), $\sigma = 0.5$, $k_1 = 2$, and $k_2 = 3$.

From these visualizations, it is observed that as the time to maturity increases, the option value generally increases, reflecting the greater potential for favorable price movements in the underlying asset before expiration. Additionally, the option value tends to increase as the asset price increases, consistent with the payoff structure of the option, which benefits from higher underlying prices. The higher volatility σ and differing liquidity parameters k_1 and k_2 influence the curvature and steepness of the option value surfaces, with Figure 4.1a exhibiting more pronounced variation due to higher volatility and liquidity parameter differences compared to Figure 4.1b.

Figure 4.2: Comparison of 2D slices of option values: (a) plots the option value $u(x,t)$ against asset price x for various fixed times t, and (b) does the same for $v(x,t)$.



(a) Option Value $u(x,t)$ for selected t



(b) Option Value $v(x,t)$ for selected t

Figures 4.2a and 4.2b show 2D line plots of the option values $u(x,t)$ and $v(x,t)$, respectively, for various fixed times to maturity t. The parameters used in these plots are the same as those used in the corresponding 3D surface plots in Figures 4.1a and 4.1b. These 2D views provide a clearer understanding of how the option values evolve with changes in the asset price and time to maturity.

4.4.2 *Subalgebra* $\mathfrak{V}_1^3 := v_2 + v_3 + v_4 = \partial_t + (\partial_u + \partial_v) + x\partial_u + x\partial_v$.

The Lagrange's system (4.31) becomes

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{1+x} = \frac{dv}{1+x},$$

there are several linear equations that can be formed from the above Lagrange's system. The first ratio is

$$\frac{dx}{0} = \frac{dt}{1}.$$

Integrating the above yields $x = C_1$, where C_1 is just a constant of integration. Hence, one of the invariants is

$$J_1 = x. \tag{4.45}$$

Checking of invariant J_1

Checking for the invariant J_1 we have

$$\begin{aligned}
\mathfrak{V}_2(J_1) &= \partial_t(J_1) + (\partial_u(J_1) + \partial_v(J_1)) + x\partial_u(J_1) + x\partial_v(J_1), \\
&= \partial_t(x) + (\partial_u(x) + \partial_v(x)) + x\partial_u(x) + x\partial_v(x), \\
&= 0.
\end{aligned}$$

This operator satisfies the invariant condition. Similarly, the integration of the equation

$$\frac{dt}{1} = \frac{du}{1+x}$$

yields $C_2 = u - t - tx$, where C_2 in the constant of integration. Thus, the second invariant is

$$J_2 = u - t - tx. \tag{4.46}$$

Checking of invariant J_2

Likewise, checking for the invariant J_2 we have

$$\begin{aligned}
\mathfrak{V}_2(J_2) &= \partial_t(J_2) + (\partial_u(J_2) + \partial_v(J_2)) + x\partial_u(J_2) + x\partial_v(J_2), \\
&= \partial_t\left(u - t - tx\right) + (\partial_u\left(u - t - tx\right) + \partial_v\left(u - t - tx\right)) \\
&+ x\partial_u\left(u - t - tx\right) + x\partial_v\left(u - t - tx\right), \\
&= -1 - x + 1 + x, \\
&= 0,
\end{aligned}$$

which also satisfies the invariant condition. Integrating the following equation

$$\frac{dt}{1} = \frac{dv}{1+x}$$

yields $C_3 = v - t - tx$, where C_3 in the constant of integration. Thus, the third invariant is

$$J_3 = v - t - tx. \tag{4.47}$$

Checking of invariant J_3

Likewise, checking for the invariant J_3 we have

$$\begin{aligned}
\mathfrak{V}_2(J_3) &= \partial_t(J_3) + (\partial_u(J_3) + \partial_v(J_3)) + x\partial_u(J_3) + x\partial_v(J_3), \\
&= \partial_t\left(v - t - tx\right) + \left(\partial_u\left(v - t - tx\right) + \partial_v\left(v - t - tx\right)\right) \\
&+ x\partial_u\left(v - t - tx\right) + x\partial_v\left(v - t - tx\right), \\
&= -1 - x + 1 + x, \\
&= 0,
\end{aligned}$$

which also satisfies the invariant condition. Designating one of the invariants as a function of the other, that is

$$J_2 = \mathcal{F}(J_1),$$

we get

$$u(x,t) = \mathcal{F}(x) + t + tx, \tag{4.48}$$

Similarly,

$$J_3 = \mathcal{G}(J_1),$$

and we obtain

$$v(x,t) = \mathcal{G}(x) + t + tx. \tag{4.49}$$

Substituting (4.48) and (4.49) in the parabolic system (4.1), we obtain

$$\begin{aligned}
\partial_t \left(\mathcal{F}(x) + t + tx\right) &= \frac{1}{2}\sigma^2 x^2 \partial_{xx}\left(\mathcal{F}(x) + t + tx\right) - ae^{(\mathcal{F}(x)+t+tx)-(\mathcal{G}(x)+t+tx)} + b, \\
\partial_t \left(\mathcal{G}(x) + t + tx\right) &= c - ce^{(\mathcal{F}(x)+t+tx)-(\mathcal{G}(x)+t+tx)}.
\end{aligned}$$

Simplifying the system of equations, we get,

$$1 + x = \frac{1}{2}\sigma^2 x^2 \mathcal{F}''(x) - ae^{\mathcal{F}(x)-\mathcal{G}(x)} + b, \tag{4.50}$$

$$1 + x = c - ce^{\mathcal{G}(x)-\mathcal{F}(x)}. \tag{4.51}$$

From (4.51) we find that

$$\mathcal{G}(x) = \mathcal{F}(x) + \ln\left(\frac{c-1-x}{c}\right) \tag{4.52}$$

Substituting (4.52) in (4.50) we obtain

$$1 + x + \frac{ac}{c-1-x} = \frac{1}{2}\sigma^2 x^2 \mathcal{F}''(x) + b \tag{4.53}$$

Upon solving the above equation, we get

$$\begin{aligned}
\mathcal{F}(x) = k_1 + xk_2 + \frac{2}{(-c+1)^2\sigma^2}\Big[&- x(-c+1)^2 + \Big((x+b-1)(c+1)^2 \\
&+ ac(x-c+1)\Big)\ln(x) + ac(-x+c-1)\ln(x-c+1)\Big].
\end{aligned} \tag{4.54}$$

plugging the above in (4.52) we get

$$\begin{aligned}\mathcal{G}(x) = k_1 + xk_2 + \frac{2}{(-c+1)^2\sigma^2}\Bigg[&- x(-c+1)^2 + \Big((x+b-1)(c+1)^2 \\ &+ ac(x-c+1)\Big)\ln(x) + ac(-x+c-1)\ln(x-c+1)\Bigg] \\ &+ \ln\left(\frac{c-1-x}{c}\right).\end{aligned} \tag{4.55}$$

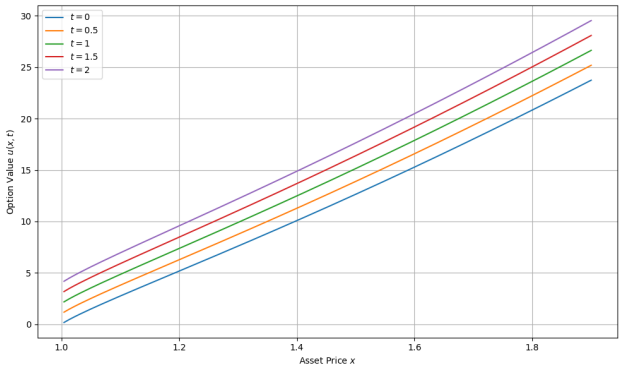
Now to find the exact invariants, we substitute (4.54), (4.55) in (4.48), (4.49) respectively and get

$$\begin{aligned}u(x,t) = k_1 + xk_2 + \frac{2}{(-c+1)^2\sigma^2}\Bigg[&- x(-c+1)^2 + \Big((x+b-1)(c+1)^2 \\ &+ ac(x-c+1)\Big)\ln(x) + ac(-x+c-1)\ln(x-c+1)\Bigg] \\ &+ t + tx.\end{aligned} \tag{4.56}$$

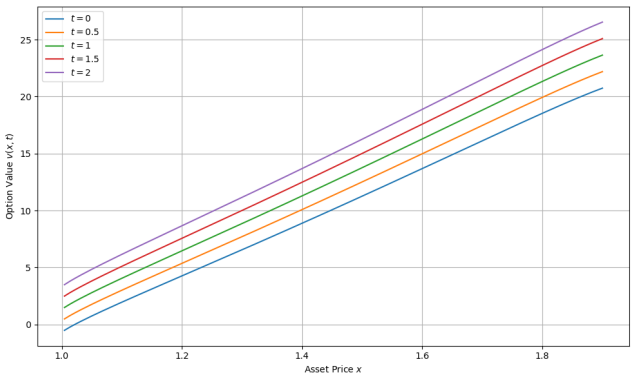
and

$$\begin{aligned}v(x,t) = k_1 + xk_2 + \frac{2}{(-c+1)^2\sigma^2}\Bigg[&- x(-c+1)^2 + \Big((x+b-1)(c+1)^2 \\ &+ ac(x-c+1)\Big)\ln(x) + ac(-x+c-1)\ln(x-c+1)\Bigg] \\ &+ \ln\left(\frac{c-x}{c}\right) + t + tx.\end{aligned} \tag{4.57}$$

Figure 4.3: Comparison of option values: (a) shows the 2D option value solution to (4.56), and (b) presents the 2D option value as a function of stock price at specific time points for (4.57).



(a) Option Value $u(x,t)$ over Stock Price at fixed time



(b) Option Value $v(x,t)$ over Stock Price at fixed time

In both Figures 4.3a and 4.3b, we used the following parameter values: $a = 1$, $b = 1$, $c = 2$ (note that $c \neq 1$ to avoid division by zero in the model equations), $\sigma = 1$, $k_1 = 1$, and $k_2 = 1$. These parameter choices help ensure numerical stability while remaining within realistic bounds of market volatility (σ) and liquidity shock effects (k_1, k_2).

In Figure 4.3b, one of the option value curves starts below zero for some stock prices. While option prices theoretically should not be negative, this behavior can arise from the selection of the parameters k_1 and k_2. In some extreme scenarios, such as those involving high volatility or liquidity shocks the pricing model might briefly show a negative adjustment, which might be seen as a cost or penalty instead of an actual tradable option price. This indicates that one should exercise caution when evaluating option values near limits or during stress situations, and it might be necessary to refine the parameters further to guarantee that option prices remain strictly non-negative.

4.4.3 *Subalgebra* $\mathfrak{V}_3 := v_1 + v_2 + v_4 = x\partial_x + \partial_t + x\partial_u + x\partial_v$.

The characteristic equation is given by,

$$\frac{dx}{x} = \frac{dt}{1} = \frac{du}{x} = \frac{dv}{x}. \tag{4.58}$$

Considering the first and second ratios from the characteristic equation (4.58), we get,

$$\frac{dx}{x} = \frac{dt}{1}. \tag{4.59}$$

Hence (4.66) implies that $c_1 = t - \ln(x)$, where c_1 is the first invariant. Next, we take the first and third ratios of (4.58),

$$\frac{dx}{x} = \frac{du}{x},$$

which results in the invariant $c_2 = u - x$, similarly considering the first and the fourth ratios,

$$\frac{dx}{x} = \frac{dv}{x},$$

the resulting invariant is $c_3 = v - x$. Thus, the invariance surface conditions $c_2 = \mathcal{W}(c_1)$ and $c_3 = \mathcal{M}(c_1)$, gives the general solutions

$$u = \mathcal{W}(t - \ln(x)) + x, \tag{4.60}$$

and

$$v = \mathcal{M}(t - \ln(x)) + x, \tag{4.61}$$

respectively. Substituting $\zeta + \ln(x)$ for t, (4.60) and (4.61) reduce to

$$u = \mathcal{W}(\zeta) + x, \tag{4.62}$$

and

$$v = \mathcal{M}(\zeta) + x. \tag{4.63}$$

Substituting (4.68) and (4.69) in the parabolic system (4.1), we get the following system of ordinary differential equations

$$2b + (-2 + \sigma^2)\mathcal{W}' + \sigma^2\mathcal{W}'' = 2ae^{\mathcal{W}-\mathcal{M}}, \tag{4.64}$$

$$c - \mathcal{M}' = ce^{\mathcal{M}-\mathcal{W}}, \tag{4.65}$$

where $\mathcal{W}' = \frac{d\mathcal{W}}{d\zeta}$, $\mathcal{W}'' = \frac{d^2\mathcal{W}}{d\zeta^2}$ and $\mathcal{M}' = \frac{d\mathcal{M}}{d\zeta}$. When writing (4.64) and (4.65) as a single ODE, we get a nonlinear ODE.

4.4.4 *Subalgebra* $\mathfrak{V}_2^1 := v_1 = x\partial_x$.

The characteristic equation is given by,

$$\frac{dx}{x} = \frac{dt}{0} = \frac{du}{0} = \frac{dv}{0}. \tag{4.66}$$

Considering the first and second ratios from the characteristic equation (4.66), we get,

$$\frac{dx}{x} = \frac{dt}{0}. \tag{4.67}$$

Hence (4.67) implies that $c_1 = t$, where c_1 is the first invariant. Next, we take the first and third ratios of (4.66),

$$\frac{dx}{x} = \frac{du}{0},$$

which results in the invariant $c_2 = u$, similarly considering the first and the fourth ratios,

$$\frac{dx}{x} = \frac{dv}{0},$$

the resulting invariant is $c_3 = v$. Thus, the invariance surface conditions $c_2 = \mathcal{D}(c_1)$ and $c_3 = \mathcal{E}(c_1)$, gives the general solutions

$$u = \mathcal{D}(t), \tag{4.68}$$

and

$$v = \mathcal{E}(t), \tag{4.69}$$

respectively. Substituting (4.68) and (4.69) in the parabolic system (4.1), we get the following system of ordinary differential equations

$$\mathcal{D}' = b - ae^{\mathcal{D}-\mathcal{E}}, \tag{4.70}$$

$$\mathcal{E}' = c - ce^{\mathcal{E}-\mathcal{D}}, \tag{4.71}$$

where $\mathcal{D}' = \frac{d\mathcal{D}}{dt}$, and $\mathcal{E}' = \frac{d\mathcal{E}}{dt}$. Making $\mathcal{D}$ the subject in (4.71), we get,

$$\mathcal{D} = \mathcal{E} + \ln\left(\frac{c}{c - \mathcal{E}'}\right). \tag{4.72}$$

Differentiating (4.72) we obtain,

$$\mathcal{D}' = \mathcal{E}' + \frac{\mathcal{E}'}{c - \mathcal{E}'}. \tag{4.73}$$

Substituting (4.72) and (4.73) in (4.70), and simplifying, we get the following ordinary differential equation,

$$b(c - \mathcal{E}') = c\mathcal{E}' - \mathcal{E}'^2 + \mathcal{E}'' - ac. \tag{4.74}$$

Upon solving the above and remembering that $v = \mathcal{E}(t)$, we deduce,

$$\mathcal{E} = v = \frac{bt + ct}{2} + k_2 - \ln\left(\cos\left(\frac{\sqrt{-b^2 + 2bc - 4ac - c^2}}{2}(t - k_1)\right)\right). \tag{4.75}$$

Differentiating (4.76) we obtain,

$$\mathcal{E}' = \frac{b + c}{2} + \frac{\sqrt{-b^2 + 2bc - 4ac - c^2}}{2}\tan\left(\frac{\sqrt{-b^2 + 2bc - 4ac - c^2}}{2}(t - k_1)\right). \tag{4.76}$$

Substituting (4.75) and (4.76) in (4.72),

$$\mathcal{D} = \frac{bt + ct}{2} + k_2 - \ln\left(\cos\left(\mathfrak{K}(t - k_1)\right)\right) + \ln\left(\frac{c}{c - (\frac{b+c}{2} + \mathfrak{K}\tan(\mathfrak{K}(t - k_1)))}\right), \tag{4.77}$$

where $\mathfrak{K} = \frac{\sqrt{-b^2+2bc-4ac-c^2}}{2}$.

4.5 Conclusion

In this chapter, we applied Lie symmetry analysis to a coupled system of partial differential equations (PDEs) designed to model the pricing of European options under liquidity shocks. This chapter demonstrated the power of Lie symmetry methods in simplifying and analyzing complex mathematical models that describe real-world financial phenomena with some limitation on highly nonlinear equations.

The identification of Lie point symmetries for the coupled parabolic system was achieved. By solving the determining equations, we uncovered the infinitesimal generators that characterizes the symmetries of the system. These symmetries provided a deeper understanding of the structural properties of the equations governing European options pricing in affected by liquidity shocks.

Through similarity reductions, we reduced the coupled PDE system into ordinary differential equations (ODEs), which are simpler to analyze and solve. The invariant solutions derived from these reductions were further explored and visualized using 2D and 3D simulations. These visualizations offered valuable insights into how option prices change when liquidity conditions shift.

5

Discussions and Conclusions

5.1 Discussion

This study focused on solving a coupled parabolic system to model European option pricing under liquidity shocks using Lie symmetry analysis. The results obtained in this thesis provide insights into the mathematical modeling of financial derivatives in imperfect markets. Below, we discuss the implications of the findings, their relation to existing literature, and the challenges encountered during the study.

5.1.1 Interpretation of Results

The identification of Lie point symmetries for the coupled parabolic system was a critical stem in understanding its structural properties. By solving the determining equations, we derived infinitesimal generators that revealed symmetries facilitated the transformation of the original partial differential equations (PDEs) into ordinary partial differential equations (ODEs) through similarity reductions.

The invariant solutions derived from these ODEs were visualized using 2D and 3D simulations, illus-

trating how European option prices evolve under varying markets conditions influenced by liquidity shocks. These simulations demonstrated that liquidity shocks significantly alter option price trajectories, deviating from the classical models like Black-Scholes, which assume constant market liquidity.

5.1.2 Comparion with Existing Literature

This research builds upon and extends prior work on option pricing under liquidity shocks, particularly the frameworks proposed by Ludkovski and Shen [53]. While their approach relied on integro-differential equations to model market states, our use of Lie symmetry analysis offers a novel perspective by providing analytical solutions to the coupled PDE system. This contrasts with traditional numerical methods, which often lack the ability to uncover invariant properties or provide closed-form solutions.

By explicitly addressing liquidity shocks, this research contributes to a growing body of literature that challenges the assumptions of classical models and highlights the importance of market dynamics in pricing financial derivatives.

5.1.3 Practical Implications

The findings have significant practical implications for financial markets:

- **Risk Management**: The model provides insights into how liquidity shocks affect option prices, enabling traders and risk managers to better anticipate and mitigate risks during periods of market instability.
- **Pricing Strategies**: The analytical solutions derived in this study can serve as benchmarks for developing more accurate pricing strategies for European options in illiquid markets.
- **Policy Development**: Regulators can use these insights to design policies aimed at minimizing systemic risks associated with sudden liquidity shocks.

5.1.4 Challenges and Limitations

While this study achieved its objectives, several challenges were encountered:

1. **Model Assumptions**: The model assumes deterministic parameters for volatility and interest rates, which may not fully capture real-world conditions where these factors are stochastic.
2. **Empirical Validation**: Due to limited available data on liquidity shocks, validating the model's predictions remains a task for future work.

Despite these challenges, the methodology demonstrated its robustness in addressing complex coupled systems and provides a foundation for further exploration.

5.2 Conclusions

This dissertation has successfully demonstrated how Lie symmetry analysis can be applied to solve a coupled parabolic system modeling European option pricing under liquidity shocks. The key contributions of this research are summarized as follows:

1. **Identification of Lie Symmetries**: The study uncovered infinitesimal generators and Lie point symmetries for the coupled PDE system, providing a deeper understanding of its structural properties.
2. **Similarity Reductions**: By employing similarity reductions, we transformed the complex PDE system into simpler ODEs, enabling analytical solutions.
3. **Invariant Solutions and Visualizations**: The invariant solutions obtained were represented through 3D simulations, offering valuable insights into option price behavior during periods of market illiquidity.

4. **Advancement in Financial Modeling**: This research complimented Lie symmetry analysis as a novel tool for addressing challenges in financial modeling caused by market heterogeneities due to liquidity shocks.

The findings underscore the utility of Lie symmetry analysis as a rich mathematical framework for solving complex systems in mathematical finance. By bridging theoretical insight with practical solutions, this research contributes to advancing both academic knowledge and industry practices.

5.2.1 Future Directions

While this study has made significant strides, there are several avenues for future research:

1. **Incorporating Stochastic Volatility**: Extending the model to include stochastic processes for volatility and interest rates could enhance its realism and applicability.
2. **Application to Other Derivatives**: The methodology could be applied to other financial instruments such as American options or exotic derivatives.
3. **Empirical Validation**: Future studies could focus on validating the model using real-world data on liquidity shocks to assess its practical effectiveness.
4. **Interdisciplinary Applications**: Similar techniques could be explored in other fields where coupled parabolic systems arise, such as economics or engineering.

In conclusion, this thesis provides a solid foundation for understanding European option pricing under liquidity shocks using Lie symmetry analysis. It encourages more research at the intersection of mathematics and finance, fostering innovative approaches to solving real-world problems in financial markets.

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