



Analytical Solutions to the Arithmetic Asian Options Pricing Model Using Lie Symmetry Methods

by

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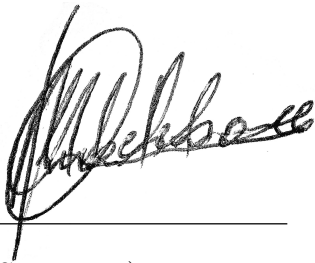
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Declaration

I, Monts'uoë Edward Kubeletsane, student number 201902922, declare that this dissertation entitled, “*Analytical Solutions to the Arithmetic Asian Options Pricing Model Using Lie Symmetry Methods*” submitted for the degree of Master of Science in Mathematics (Applied) at the National University of Lesotho is my original work and any work done by others has been acknowledged accordingly.



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Abstract

This dissertation explores the use of Lie symmetry methods to find analytical solutions for arithmetic Asian options; path-dependent financial derivatives widely used for risk management in commodity markets. The pricing problem is formulated as a partial differential equation (PDE) involving the asset price, time, and the running average of the asset price. Lie symmetry analysis is applied to the PDE to compute its infinitesimal generators, determine an optimal system of one-dimensional sub-algebras, and perform symmetry reductions. Each reduced PDE obtained through this process also admits further symmetries, allowing for successive reductions and the construction of exact invariant solutions. Techniques such as Riccati reductions and the Frobenius method are employed to solve the resulting ordinary differential equations (ODEs). The study further examines the influence of key financial parameters (volatility, interest rate, and time to maturity) on the structure and behavior of the pricing solutions. The findings contribute to the theoretical understanding of Asian option pricing and provide analytical benchmarks for validating numerical approaches.

Keywords

Arithmetic Asian options;

Option value (payoff);

Financial derivatives;

Lie symmetry method;

Optimal system;

Invariant solutions;

Partial differential equations.

Dedication

To my family, friends, and well-wishers.

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List of Abbreviations

PDE(s), Partial Differential Equation(s);

ODE(s), Ordinary Differential Equation(s);

DE(s), Differential Equation(s);

Eqn(s), Equation(s);

i.e, That is;

w.r.t, with respect to.

CHAPTER 1

GENERAL INTRODUCTION

1.1 Introduction

Asian options were first introduced by Ingersoll [1] and were initially issued in 1987 when the Bankers Trust Tokyo office used them to price average options on crude oil contracts. Hence, these derivatives are called “Asian options” [2]. Today, they are commonly traded on currencies and commodity products [3, 4]. The averaging feature of Asian options reduces volatility compared to their underlying assets, making them more cost-effective than standard options on the same assets [5].

The payoff of an Asian option depends on the arithmetic or geometric average price of the underlying asset, such as stocks, commodities, or currencies. In practice, the averaging is often done on specific dates, but in the context of mathematical modeling, it is often assumed that the average is taken continuously for simplicity and accuracy. The continuously

averaged option is considered an approximation of the discrete case [4], allowing for more straightforward analysis and pricing.

In this study, we will only focus on the arithmetic Asian options, also known as *average options*, which are derivatives whose valuation depends on the average price of an underlying asset over a defined period (See [6, 7, 8]). *Arithmetic* describes the process that is employed to determine the average price. It specifically entails adding up all of the underlying asset's prices throughout the specified period and dividing the result by the total number of prices. As a consequence, an average price is calculated with equal weights for every observation [9].

Arithmetic Asian options play a vital role in managing risk in commodity markets, particularly those influenced by commodities like crude oil, agricultural products [10], and precious metals. Multinational corporations, such as those in the food industry relying on commodities like wheat, can utilize these options to hedge against price fluctuations, ensuring stable production costs [11, 12].

Moreover, arithmetic Asian options have broader implications, impacting consumer prices and household budgets [13]. For instance, fluctuations in commodity prices, such as crude oil [14, 15], directly affect everyday expenses like transportation costs. Thus, airlines can employ these options to mitigate fuel price risks and maintain stable ticket prices. Additionally, in the realm of precious metals like gold, these options enable investors to capitalize on long-term price trends while mitigating short-term volatility risks [16]. Jewelry manufacturers also leverage arithmetic Asian options to manage raw material costs, ensuring consistent pricing for consumers despite fluctuating gold prices.

In summary, arithmetic Asian options are essential for stabilizing costs in commodity markets,

demanding precise pricing models. They are worth studying because their path-dependent nature introduces analytical challenges not present in standard options, requiring specialized techniques for accurate valuation. By developing reliable pricing models, these options enable effective risk management for firms, help stabilize consumer prices, and contribute to more resilient financial and commodity markets.

1.2 Background to the Study

This section is divided into three subsections, with the first subsection focusing on *Lie symmetries*. The discussion highlights the historical development of Lie symmetries and its applications in various fields, including its extensive use in option pricing models within financial markets. The second subsection focuses on *financial markets*, explaining the significance of pricing models and options in financial decision-making, risk management, and investment strategies. Last subsection is on *arithmetic Asian options PDE*, showcasing the work done by other research on the governing PDE.

1.2.1 Lie Symmetries

Lie symmetries, which involve identifying transformations that leave differential equations invariant, provide a systematic approach to reducing and solving DEs. This framework was pioneered by Sophus Lie (1842–1899) in the late 19th century amid key advances in geometry and algebra [17, 18, 19]. Influenced by Galois theory through Peter Ludvig Sylow’s (1862) lectures and the geometric insights of Jean Poncelet and Julius Plücker, Lie developed continuous transformation groups to simplify DEs [17]. His (1874) theorem demonstrated that one-parameter groups preserving a DE’s form yield exact solutions or reductions, introducing Lie algebras as infinitesimal generators [18]. In collaboration with Friedrich Engel, he systematized these ideas in *Theorie der Transformationsgruppen (1888–1893)*, drawing further inspiration from Felix Klein’s Erlangen Program and Camille Jordan’s group theory [17].

In the early 20th century, Élie-Joseph Cartan (1869–1951) advanced Lie’s framework by classifying simple Lie algebras (1910–1914), refining Wilhelm Killing’s earlier structure theory, and applying symmetry methods to differential geometry and systems of PDEs [20, 21, 22]. His contributions to Pfaffian systems and equivalence problems deepened the algebraic and geometric foundations of Lie theory [17]. Although Lie’s geometric intuition initially met with resistance, Engel’s expositions and Cartan’s rigor established Lie symmetries as central to mathematical physics, integrability theory, and conservation laws [19].

Since then, Lie symmetry analysis has evolved into a powerful method for solving nonlinear DEs in various disciplines, motivated by the demand for analytical solutions to complex systems. In physics, it uncovers conservation laws and integrable structures [23, 24, 25]; in engineering, it simplifies models in continuum mechanics and elasticity [26, 27]. In mathematical finance, Lie symmetries facilitate analytical solutions to pricing models, including the Black-Scholes PDE and its nonlinear or fractional extensions, enhancing both risk management and computational efficiency [28, 29, 30, 31].

Gazizov and Ibragimov were among the first to apply Lie symmetries to financial PDEs; in 1998, they transformed the Black-Scholes equation into the heat equation, obtaining closed-form solutions for European options [32, 33]. Lo et al. extended this approach, using Lie algebras to price multi-asset and exotic options under the constant elasticity of variance (CEV) model with time-dependent parameters [34, 35]. Björk and Landén (2002) applied Lie algebras to interest rate models, enabling bond option pricing [36]. Recent work by Yue and Shen (2023–2024) on fractal bond-pricing models and Ramoetsi (2021–2022) on fractional Black-Scholes equations builds on these foundations to address modern market complexities [37, 38].

Additionally, Kaibe and O’Hara [39] applied Lie symmetry techniques to interest rate derivatives, deriving exact solutions for zero-coupon bond pricing models based on Björk’s Lie algebra methods. Kaibe’s PhD thesis [40] develops a comprehensive framework for symmetry-based solutions to interest rate models such as the Vasicek equation, addressing challenges like negative interest rates and providing analytical benchmarks. These contributions highlight Lie symmetry analysis as a versatile tool in modern financial mathematics, bridging deep theoretical insights with practical applications.

1.2.2 Financial Markets

Pricing and options are central to financial markets, driving investment strategies, risk management, and market efficiency amid rising complexity since the 1990s [41, 42]. The rapid growth of derivatives markets and increased volatility have underscored the need for robust pricing frameworks and analytical tools to address these dynamics [5].

Pricing

Asset pricing seeks to determine the fair value of financial instruments by assessing their risk and expected return. The Capital Asset Pricing Model (CAPM), developed by Sharpe [43], Lintner [44], and Treynor [45], uses a single market risk factor (beta) to estimate returns. Its parsimonious design, while theoretically appealing, struggles with firm-specific factors and out-of-sample performance [46]. The Arbitrage Pricing Theory (APT), introduced by Ross [47, 48], employs multiple factors but faces challenges in factor selection [49]. Since the 1990s, factor-based models like the Fama-French five-factor model, incorporating size, value, profitability, and investment [50], and the Carhart four-factor model, adding momentum [51, 52], have improved explanatory power at the cost of complexity, spurring interest in dynamic and machine learning-based approaches [53].

Options

Options, key derivatives, grant the right to buy (call) or sell (put) an underlying asset at a strike price (K) by expiration. Essential for portfolio optimization, hedging, and speculation, they enable investors to capitalize on upward (long calls) or downward (long puts) price movements while mitigating risks through short positions [54, 55, 56]. Option values hinge on the underlying asset price (S), volatility (σ), time to expiration, and risk-free rate (r). The Black-Scholes-Merton model [57], assuming constant volatility and log-normal prices, prices European options via the PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad V = V(S, t), \quad (1.1)$$

though its constant volatility assumption limits real-world applicability [5].

Exotic options, including Asian, Barrier (common in forex exchange markets), Bermudan, and Lookback variants, introduce path-dependency, complicating valuation. This dissertation targets arithmetic Asian options, whose payoff depends on the average asset price $y = \frac{1}{T} \int_0^T S_u du$, widely used in commodity markets for smoothing price volatility [58]. Pricing methods include Black-Scholes adaptations [8], Monte Carlo simulations [59, 60], and perturbation techniques [2], but these often rely on numerical approximations.

1.2.3 Arithmetic Asian Options PDE

With no general closed-form solution available for the arithmetic Asian options PDE, researchers have relied primarily on PDE-based formulations, numerical schemes, asymptotic analysis, and symmetry methods. Although Lie symmetry techniques have been applied to the governing PDE, prior studies typically halt at a single reduction or rely on Laplace or

Fourier transforms together with numerical evaluation. The pricing PDE,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S + SV_A - rV = 0, \tag{1.2}$$

remains analytically intractable, and this has motivated the development of diverse numerical, asymptotic, and symmetry-based approaches [61, 62, 63, 28, 64, 65].

Numerical approaches have been central. Zvan, Forsyth, and Vetzal [61] introduced finite-volume schemes with flux limiters to stabilise the convection–diffusion structure of the PDE, suppressing oscillations that arise under low volatility and American-style features. While effective, their method was computationally demanding on two-dimensional grids. Later, Mudzimbabwe, Patidar, and Witbooi [65] applied Crank–Nicolson discretisations to a homogeneity-reduced PDE, demonstrating stable convergence and improved efficiency over Monte Carlo benchmarks, though their results were limited to European-style options with continuous averaging. Asymptotic and transform-based work provided complementary insights: Dewynne and Shaw [62] obtained one-dimensional similarity reductions and asymptotic “Black–Scholes-like” formulae valid for low volatility, while Elshegmani et al. [63] transformed the PDE into a constant-coefficient heat equation but without delivering explicit solutions or validation.

A further stream has focused on Lie symmetries. Caister, Govinder, and O’Hara [28] reduced the PDE to two variables using symmetry methods and solved the reduced equations via Laplace transforms and special functions. Extending this, Caister et al. [64] developed an optimal system of symmetries that unified earlier ad hoc reductions such as Rogers–Shi and Věčer, yielding new invariant families but still relying on transform methods and encountering boundary condition difficulties.

Despite significant progress in the analysis of arithmetic Asian options, no existing study has

fully resolved the pricing PDE through a complete double reduction using Lie symmetries alone. This dissertation addresses that gap by performing a full, exact double reduction of the two-state PDE to a system of ODEs via Lie symmetry techniques. The resulting invariant solutions are explicit, analytically derived, and constitute a novel class not previously documented in the literature.

Building on this theoretical advance, the model is formulated as a two-state¹ PDE from [65]:

$$-rv + v_t + rxv_x + \frac{1}{2}\sigma^2x^2v_{xx} + xv_y = 0, \quad v = v(t, x, y), \quad (1.3)$$

where $x = S_t$ is the asset price and y the running average, extending the classical Black–Scholes–Merton PDE (1.1) to account for path-dependency. The PDE, (1.3) is subject to:

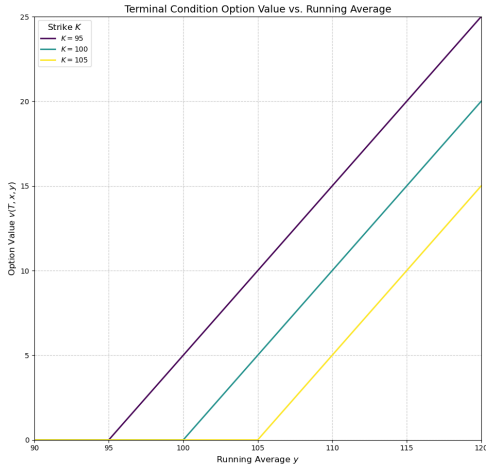
1. **Terminal Condition (TC):** at maturity ($t = T$), the option value reflects the payoff function:

$$v(T, x, y) = h(y) = \max(y - K, 0), \quad x \geq 0, y \in \mathbb{R}, \quad (1.4)$$

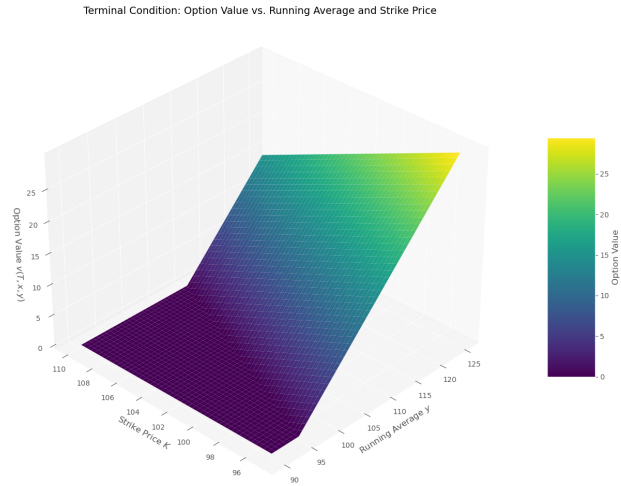
where K is the strike price, and $\max(y - K, 0)$ denotes the positive part of $(y - K)$, representing the intrinsic value of the option. The payoff structure follows that of a standard call option:

- in-the-money (ITM): when $y > K$, the option has value, and the payoff is $y - K$,
- out-of-the-money (OTM): when $y < K$, the payoff remains zero, as the option is worthless.

¹The term *two-state* refers to the fact that the model depends on two independent state variables: the current asset price x and the running average y . This structure reflects the path-dependent nature of arithmetic Asian options, where both the spot price and its time-averaged value influence the option's value.



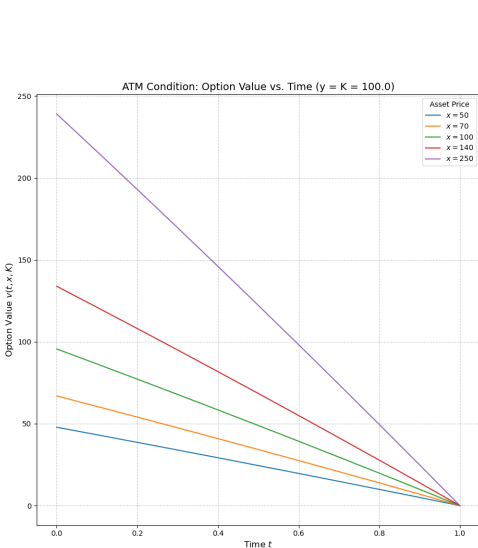
(a) 2D-Payoff graph.



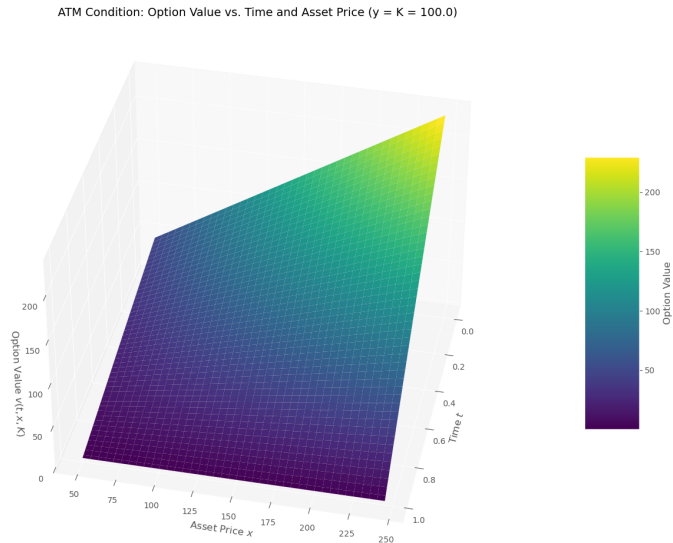
(b) 3D-Payoff graph.

Figure 1.1: 2D and 3D Option value (payoff) graphs under TC; for $90 \leq y \leq 120$ and $95 \leq K \leq 105$.

2. **At-the-money (ATM) Condition:** when $y = K$, the payoff is zero, indicating no intrinsic value at the break-even point. This introduces a new condition $v(t, x, K) = \frac{1}{Tr} (1 - e^{-r(T-t)}) x$, showing that the option's value depends on the remaining time as well as the underlying asset's price.



(a) 2D-Payoff graph.



(b) 3D-Payoff graph.

Figure 1.2: 2D and 3D Option value (payoff) graphs under ATM condition; for $50 \leq x \leq 250$, $0 \leq t \leq 1$ and $y = K = 100$.

3. **Zero Price Condition (Z-P Condition)**: for a zero asset price ($x = 0$), the option value is discounted based on the remaining time to maturity. This condition is given by:

$$v(t, 0, y) = e^{-r(T-t)} \max(y - K, 0), \quad 0 \leq t \leq T, y \in \mathbb{R}. \quad (1.5)$$

Here, r represents the risk-free rate, and the factor $e^{-r(T-t)}$ accounts for the time value of money. Similar to the TC, we have the following cases to be considered:

- when $y > K$, the option retains value, determined by the discounted payoff $e^{-r(T-t)}(y - K)$,
- when $y \leq K$, the option is worthless, as the payoff is zero.

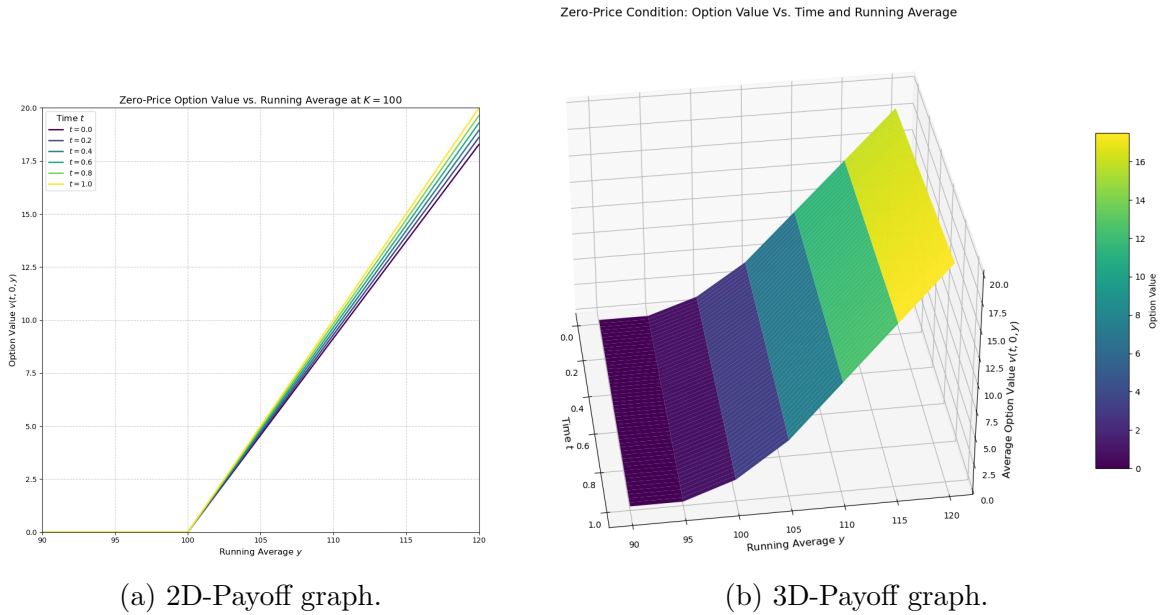


Figure 1.3: 2D and 3D Option value (payoff) graphs under Z-P condition; for $90 \leq y \leq 120$ and $0 \leq t \leq 1$.

Part of this work has been submitted for publication under the title “*Optimal System and Invariant Solutions from Reduced ODEs Resulting from the Arithmetic Asian Options PDE*” and is currently under review.

1.3 Derivation of the Arithmetic Asian Options Model

In this section, we will give a summary of the derivation of the arithmetic Asian options model described by a two state PDE (1.3), as well as its boundary conditions. The approach follows several steps, from modeling the underlying asset dynamics to applying Itô's Lemma and deriving the PDE (1.3).

1.3.1 Introduction

According to the Fundamental Theorem of Asset Pricing (refer to, e.g., [66]), the value of a fixed strike Asian call option is expressed as (see [59, 65]):

$$C(t) = \mathbb{E}_t^Q \left[e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u du - K \right)^+ \right], \quad (1.6)$$

where:

- $C(t)$ is the price of the Asian call option,
- r is the risk-free interest rate,
- S_t is the price of the underlying asset at time t ,
- K is the strike price,
- \mathbb{E}_t^Q is the expectation under the risk-neutral measure Q .

Our aim is to derive a PDE that models the price of the option as a function of the asset price S_t , its running average y_t , and time t .

1.3.2 Dynamics of the Underlying Asset

Let S_t represent the price of the underlying asset at time t . Under the risk-neutral measure Q , we assume the asset price follows a Geometric Brownian Motion (GBM). This is a standard

assumption in financial modeling. The dynamics of the asset price under Q are governed by the following Stochastic differential equation(SDE):

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad (1.7)$$

where:

- r is the risk-free interest rate,
- σ is the volatility of the asset,
- W_t^Q is a Wiener process under the risk-neutral measure Q .

This equation describes how the asset price evolves over time. The term $rS_t dt$ reflects the drift of the asset price, while $\sigma S_t dW_t^Q$ captures the randomness due to market fluctuations.

1.3.3 Running Average of the Asset Price

The payoff of an arithmetic Asian option depends on the running integral y_T of the asset price, defined as:

$$y_T = \int_0^T S_u du.$$

At time t , the running integral is:

$$y_t = \int_0^t S_u du.$$

Thus, at maturity T , we have:

$$y_T = y_t + \int_t^T S_u du. \quad (1.8)$$

1.3.4 Payoff Function and Discounted Price

The payoff of an arithmetic Asian call option depends on the running integral of the asset price at maturity. For an arithmetic Asian call option with strike price K , the payoff is given

by:

$$h(y_T) = (y_T - K)^+, \quad (1.9)$$

where y_T is the running integral at maturity and K is the strike price. The term $(\cdot)^+$ denotes the positive part, meaning the option only has value if the average price exceeds the strike price.

The undiscounted price of the option at time t is given by the risk-neutral expectation:

$$u(t, x, y) = \mathbb{E}_t^Q [h(y_T)], \quad (1.10)$$

where $u(t, x, y)$ is the undiscounted option price at time t , with $x = S_t$ and $y = y_t$.

The discounted price of the option at time t , denoted by $v(t, x, y)$, is related to the undiscounted price by:

$$v(t, x, y) = e^{-r(T-t)} u(t, x, y), \quad (1.11)$$

where $e^{-r(T-t)}$ is the discount factor due to the risk-free interest rate r .

1.3.5 Applying Itô's Lemma

We apply *Itô's Lemma* to the function $u(t, x, y)$, where $x = S_t$ and $y = y_t$. The dynamics of the variables are:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q, \quad (1.12)$$

and

$$dy_t = S_t dt = x dt. \quad (1.13)$$

Applying Itô's Lemma to $u(t, x, y)$, we get:

$$du = u_t dt + u_x dS_t + u_y dy_t + \frac{1}{2} u_{xx} (dS_t)^2. \quad (1.14)$$

Substitute the expressions for dS_t and dy_t :

$$du = u_t dt + u_x(rS_t dt + \sigma S_t dW_t^Q) + u_y S_t dt + \frac{1}{2} u_{xx} \sigma^2 S_t^2 dt. \quad (1.15)$$

Simplifying:

$$du = \left(u_t + rxu_x + xu_y + \frac{1}{2} \sigma^2 x^2 u_{xx} \right) dt + \sigma xu_x dW_t^Q. \quad (1.16)$$

1.3.6 Martingale Property

Under the risk-neutral measure Q , the undiscounted price $u(t, S_t, y_t)$ is a martingale because it is the conditional expectation of the payoff $h(y_T)$ with no discounting in the expectation (see (1.10)). For a process to be a martingale, its drift term (the coefficient of dt) must be zero. From (1.16), the drift term is:

$$u_t + rxu_x + xu_y + \frac{1}{2} \sigma^2 x^2 u_{xx}.$$

Setting this to zero gives the PDE for the undiscounted price:

$$u_t + rxu_x + xu_y + \frac{1}{2} \sigma^2 x^2 u_{xx} = 0. \quad (1.17)$$

1.3.7 Final Transformation to the Discounted Price

Recall that the discounted option price (1.11) is related to the undiscounted option price (1.17) by:

$$v(t, x, y) = e^{-r(T-t)} u(t, x, y).$$

To convert the PDE for $u(t, x, y)$ into one for $v(t, x, y)$, we use the following relations (obtained using the chain rule for derivatives and $u = e^{r(T-t)}v$):

$$u_t = e^{r(T-t)}(v_t - rv), \quad u_x = e^{r(T-t)}v_x, \quad u_{xx} = e^{r(T-t)}v_{xx}, \quad u_y = e^{r(T-t)}v_y.$$

Substituting these into PDE (1.17) for $u(t, x, y)$, we get:

$$-rv + v_t + rxv_x + xv_y + \frac{1}{2}\sigma^2x^2v_{xx} = 0,$$

which is the final PDE for the price of the arithmetic Asian option, subject to the boundary conditions:

$$v(T, x, y) = h(y) = (y - K)^+, \quad x \geq 0, \quad y \in \mathbb{R}, \quad (1.18)$$

$$v(t, 0, y) = e^{-r(T-t)}(y - K)^+, \quad 0 \leq t \leq T, \quad y \in \mathbb{R}. \quad (1.19)$$

1.4 Aim and Objectives

The primary aim of this dissertation is to obtain analytical solutions to the PDE (1.3) using Lie symmetry methods. This goal will be accomplished through the following objectives:

1. Find the determining equations.
2. Solve the determining equations to obtain the infinitesimal generators.
3. Construct an optimal system of one-dimensional sub-algebras for systematic reduction.
4. Use the optimal system to reduce the PDE to ODEs.
5. Solve the reduced ODEs to obtain exact invariant solutions.
6. Investigate how changes in parameters such as volatility, interest rates, and time influence option prices.

1.5 Thesis Outline

The structure of this dissertation is as follows:

- In Chapter 2, we outline the general procedure followed to obtain analytical solutions to the governing PDE. This includes a step-by-step framework involving the identification of symmetries, construction of reduced equations, and solution methods used throughout the study.
- Chapter 3 presents the application of the Lie symmetry methods to the model PDE. The Lie point symmetries admitted by the equation are systematically determined.
- In Chapter 4, we construct an optimal system of one-dimensional sub-algebras using Lie brackets and adjoint representation tables. These are then employed to reduce the PDE to ODEs, which are subsequently solved to obtain invariant solutions.
- Chapter 5 is devoted to the presentation and discussion of the results. Both the mathematical properties and economic interpretations of the solutions are examined in detail.
- Finally, Chapter 6 offers general discussions and concluding remarks, summarizing the key findings and suggesting directions for future work.

SYMMETRIES OF DIFFERENTIAL EQUATIONS

2.1 Introduction

Many mathematical models have symmetries, especially those that are expressed in terms of DEs [67, 68, 69, 70]. Lie group theory, named after the Norwegian mathematician Marius Sophus Lie, is the branch of mathematics that represents and synthesizes the symmetries of DEs [71, 72, 73]. The Lie symmetry method provides a structured approach to deriving exact solutions for both PDEs and ODEs [74, 75, 76, 77, 78, 79, 80].

In this chapter, we begin by introducing the key concepts and definitions underlying the Lie symmetry approach, including the structure of differential equations, symmetry transformations, and infinitesimal generators. We then present Lie's algorithm as a systematic procedure for finding symmetries. This is followed by a brief discussion on Lie algebras, including the Lie bracket and its properties. Finally, we explore the notion of invariant so-

lutions, which form the basis for reducing complex differential equations to simpler forms through symmetry-based methods.

The contents of this chapter are adopted from [81, 82, 83, 84]

2.2 Preliminaries

In this section, we give some important definitions.

Definition 2.2.0.1. A k^{th} -order ($k \geq 1$) system E of s DEs is defined by

$$E^\sigma(x, u, u^{(1)}, \dots, u^{(k)}) = 0, \quad \sigma = 1, \dots, s, \quad (2.1)$$

where $u \equiv (u^1, u^2, \dots, u^q)$ is the dependent vector, $x \equiv (x^1, x^2, \dots, x^n)$ is the independent vector, and $u^{(1)}, u^{(2)}, \dots, u^{(k)}$ are respectively the collection of all first, second, up to k th-order derivatives.

Definition 2.2.0.2. A symmetry transformation of the system (2.1) is an invertible transformation of the variables x and u , namely

$$\bar{x}^i = f^i(x, u), \quad \bar{u}^\alpha = \phi^\alpha(x, u), \quad i = 1, \dots, n; \quad \alpha = 1, \dots, q, \quad (2.2)$$

that leaves (2.1) form-invariant in the new variables \bar{x} and \bar{u} , i.e.,

$$E^\sigma(\bar{x}, \bar{u}, \bar{u}^{(1)}, \dots, \bar{u}^{(k)}) = 0, \quad \sigma = 1, \dots, s, \quad (2.3)$$

whenever (2.1) is satisfied.

Definition 2.2.0.3. A set G of transformations

$$T_a : \bar{x}^i = f^i(x, u, a), \quad \bar{u}^\alpha = \phi^\alpha(x, u, a), \quad i = 1, \dots, n; \quad \alpha = 1, \dots, q, \quad (2.4)$$

is called a continuous one-parameter (local) Lie-group of transformations in \mathbb{R}^{n+q} provided the group properties of closure, identity, and inverses are satisfied. Here f^i and ϕ^α are differentiable functions and a is a real parameter which continuously takes values in a neighborhood $D \subseteq \mathbb{R}$ of $a = 0$.

Definition 2.2.0.4. An infinitesimal generator X of the group transformations G (2.4) is the differential operator of the form

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}, \quad (2.5)$$

such that

$$\begin{aligned} \bar{x}^i &= x^i + a\xi^i(x, u) + O(a^2) = (1 + aX)x^i, \\ \bar{u}^\alpha &= u^\alpha + a\eta^\alpha(x, u) + O(a^2) = (1 + aX)u^\alpha. \end{aligned} \quad (2.6)$$

Here and throughout this section, the Einstein summation convention is adopted. The one-parameter group elements (2.6) are known as the *infinitesimal transformations* obtained from (2.4) by first-order (Taylor expansion) approximations around parameter $a = 0$. The operator (2.5) is also called the Lie point symmetry generator.

Definition 2.2.0.5. The extended infinitesimal generator $X^{[k]}$ of the k th prolonged (extended) group $G^{[k]}$ on the space $(x, u, \dots, u^{(k)})$ is called the k^{th} prolongation of X , given by

$$\begin{aligned} X^{[k]} &= \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u^{(1)}) \frac{\partial}{\partial u_i^\alpha} \\ &+ \cdots + \zeta_{(i_1 \dots i_k)}^\alpha \frac{\partial}{\partial u_{(i_1 \dots i_k)}^\alpha}. \end{aligned} \quad (2.7)$$

The coefficients ζ_s are defined recursively by the prolongation formulae

$$\begin{aligned}
\zeta_i^\alpha &= D_i(\eta^\alpha) - u_{(j)}^\alpha D_i(\xi^j), \\
\zeta_\alpha^{ij} &= D_j(\zeta_i^\alpha) - u_{(il)}^\alpha D_j(\xi^l), \\
&\vdots \\
\zeta_{i_1 \dots i_k}^\alpha &= D_{i_k}(\zeta_{i_1 \dots i_{k-1}}^\alpha) - u_{(i_1 \dots i_k l)}^\alpha D_{i_k}(\xi^l),
\end{aligned} \tag{2.8}$$

where

$$D_j = \frac{\partial}{\partial x^j} + u_{(j)}^\alpha \frac{\partial}{\partial u^\alpha} + u_{(jk)}^\alpha \frac{\partial}{\partial u_{(k)}^\alpha} \dots; u_{(j)}^\alpha = D_j(u^\alpha), u_{(jk)}^\alpha = D_j(u_{(k)}^\alpha) \tag{2.9}$$

is the total derivative operator with respect to x^i .

2.3 Lie's Algorithm

In this section we introduce the necessary steps to be followed when calculating point symmetries of DEs.

1. Write E given by (2.1) (the PDE or ODE being solved, such as (1.3)) such that all the terms are on the left-hand side.
2. Write the generator of symmetry

$$X = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \tag{2.10}$$

3. Prolong the symmetry generator X to the order which is the same as that of E , i.e.,

$$\begin{aligned}
X^{[k]} &= \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta^\alpha(x, u) \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u^{(1)}) \frac{\partial}{\partial u_i^\alpha} \\
&+ \dots + \zeta_{(i_1 \dots i_k)}^\alpha \frac{\partial}{\partial u_{(i_1 \dots i_k)}^\alpha}.
\end{aligned} \tag{2.11}$$

where the variables ζ_i^α are given by (2.8).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated on the surface (2.1) yielding the symmetry conditions

$$X^{[k]} (E^\sigma(x, u, u_{(1)}, \dots, u_{(k)})) |_{(2.1)} = 0, \quad \sigma = 1, \dots, s. \quad (2.12)$$

5. Substitute the ζ_i^α upon expansion of the symmetry conditions and replace the derivatives which are to be eliminated.
6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDEs in terms of ξ^i and η^α .
7. Solve the over-determined system for the infinitesimals ξ^i and η^α to obtain symmetries of E .
8. Construct one-parameter groups.

2.4 Lie Algebra

Definition 2.4.0.1. A Lie algebra is a vector space L over a field \mathbb{F} with a binary operation $[-, -] : L \times L \rightarrow L$ called Lie bracket (also known as commutator), such that the following axioms are satisfied:

- (i) Bilinearity: If $X_1, X_2, X_3 \in L$ and $a, b \in \mathbb{F}$, then

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3].$$

- (ii) Skew-Symmetry: If $X_1 \in L$, then

$$[X_1, X_1] = 0,$$

and this implies that, for all $X_1, X_2 \in L$,

$$[X_1, X_2] = -[X_2, X_1].$$

(iii) Jacobi Identity: If $X_1, X_2, X_3 \in L$, then

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

Definition 2.4.0.2. Consider a Lie algebra L . If the vector space L is finite-dimensional, its dimension is the dimension of the Lie algebra, i.e, the finite-dimensional Lie algebra of dimension r is denoted by L_r .

In this dissertation we take \mathbb{F} to be the field of real numbers \mathbb{R} . We define the Lie Bracket $[-, -]$ on the set of vector field \mathcal{V} as

$$[X_1, X_2] = X_1X_2 - X_2X_1 \quad \text{for any } X_1, X_2 \in \mathcal{V}, \quad (2.13)$$

where

$$X_1 = \xi_1^i(x, u) \frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u) \frac{\partial}{\partial u^\alpha} \quad (2.14)$$

and

$$X_2 = \xi_2^i(x, u) \frac{\partial}{\partial x^i} + \eta_2^\alpha(x, u) \frac{\partial}{\partial u^\alpha}. \quad (2.15)$$

The binary operation (2.13) makes the space of vector field \mathcal{V} a Lie algebra.

2.5 Invariant Solutions

The primary motivation for determining the symmetries of DEs is to use them to uncover the structure of the solution space. A notable characteristic of a symmetry is its ability to

transform a solution into a different one. When dealing with PDEs, finding a general solution can be extremely challenging or even impossible. Therefore, it is often necessary to rely on particular solutions. Among these, invariant solutions can be systematically identified when the symmetries of the underlying equation are known.

Definition 2.5.0.1. A solution $u^\alpha = F^\alpha(x^1, x^2, \dots, x^n)$ of E^α is invariant under the one-parameter group of transformations if

$$\bar{u}^\alpha = F^\alpha(\bar{x}^1, \bar{x}^2, \dots, \bar{x}^n). \quad (2.16)$$

If we perform the first-order Taylor expansion of (2.16) around $a = 0$ and use the first-order approximations of the one-parameter group we obtain

$$u^\alpha + a\eta^\alpha + \dots = F^\alpha + aXF^\alpha + \dots \quad (2.17)$$

whenever $u = F$. This equation implies that

$$\eta^\alpha = XF^\alpha \quad (2.18)$$

whenever $u = F$. Therefore,

$$u = F(x^1, \dots, x^n), \quad (2.19)$$

is invariant under transformations provided

$$X(u^\alpha - F^\alpha)|_{(2.19)} = 0. \quad (2.20)$$

Conversely, it can be shown that if (2.20) is satisfied, then $u = F$ is invariant. Equation (2.20) represents a system of first-order quasi-linear PDEs that can be solved to obtain

the functional form of F . Once this form is obtained, it is substituted back into the original system, leading to PDEs with fewer independent variables. If the symmetries of these reduced equations are known, further reductions can be performed. In favorable cases, this process results in closed-form solutions of the original system.

APPLICATION OF LIE SYMMETRY METHOD

3.1 Finding the Determining Equations

The infinitesimal generator for eqn (1.3) w.r.t the generator (2.5) is given by

$$X = \xi^1(t, x, y, v) \partial_t + \xi^2(t, x, y, v) \partial_x + \xi^3(t, x, y, v) \partial_y + \eta(t, x, y, v) \partial_v. \quad (3.1)$$

The second prolongation using eqn (2.7) is given by

$$\begin{aligned} X^{[2]} = & \xi^1 \partial_t + \xi^2 \partial_x + \xi^3 \partial_y + \eta \partial_v + \zeta_t \partial_{v_t} + \zeta_x \partial_{v_x} + \zeta_y \partial_{v_y} \\ & + \zeta_{tt} \partial_{v_{tt}} + \zeta_{tx} \partial_{v_{tx}} + \zeta_{ty} \partial_{v_{ty}} + \zeta_{xx} \partial_{v_{xx}} + \zeta_{xy} \partial_{v_{xy}} + \zeta_{yy} \partial_{v_{yy}}. \end{aligned} \quad (3.2)$$

Equation (3.1) is the infinitesimal generator of the model PDE (1.3) if and only if

$$X^{[2]} \left(-rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} + xv_y \right) |_{(1.3)} = 0. \quad (3.3)$$

From eqns (3.2) and (3.3), we have that

$$\xi^2 (rv_x + \sigma^2 xv_{xx} + v_y) - r\eta + \zeta_t + rx\zeta_x + x\zeta_y + \frac{1}{2}\sigma^2 x^2 \zeta_{xx} = 0, \quad (3.4)$$

which simplifies to

$$2\xi^2 (rv_x + \sigma^2 xv_{xx} + v_y) - 2r\eta + 2\zeta_t + 2rx\zeta_x + 2x\zeta_y + \sigma^2 x^2 \zeta_{xx} = 0. \quad (3.5)$$

The total derivatives w.r.t t, x, y are given as follows, respectively (With the use of (2.9)):

$$D_t = \frac{\partial}{\partial t} + v_t \frac{\partial}{\partial v} + v_{tt} \frac{\partial}{\partial v_t} + v_{tx} \frac{\partial}{\partial v_x} + v_{ty} \frac{\partial}{\partial v_y} + \dots, \quad (3.6a)$$

$$D_x = \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v} + v_{tx} \frac{\partial}{\partial v_t} + v_{xx} \frac{\partial}{\partial v_x} + v_{xy} \frac{\partial}{\partial v_y} + \dots, \quad (3.6b)$$

$$D_y = \frac{\partial}{\partial y} + v_y \frac{\partial}{\partial v} + v_{ty} \frac{\partial}{\partial v_t} + v_{xy} \frac{\partial}{\partial v_x} + v_{yy} \frac{\partial}{\partial v_y} + \dots \quad (3.6c)$$

When considering eqn (2.8), the ζ_s are given as follows:

$$\zeta_t = D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2) - v_y D_t(\xi^3), \quad (3.7a)$$

$$\zeta_x = D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2) - v_y D_x(\xi^3), \quad (3.7b)$$

$$\zeta_y = D_y(\eta) - v_t D_y(\xi^1) - v_x D_y(\xi^2) - v_y D_y(\xi^3), \quad (3.7c)$$

$$\zeta_{xx} = D_x(\zeta_x) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2) - v_{xy} D_x(\xi^3). \quad (3.7d)$$

Now, from eqns (3.6a) and (3.7a), we have that

$$\zeta_t = \eta_t + v_t(\eta_v - \xi_t^1) - v_t^2 \xi_v^1 - v_x \xi_t^2 - v_t v_x \xi_v^2 - v_y \xi_t^3 - v_t v_y \xi_v^3. \quad (3.8)$$

From eqns (3.6b) and (3.7b) we have that

$$\zeta_x = \eta_x + v_x(\eta_v - \xi_x^2) - v_x^2 \xi_v^2 - v_t \xi_x^1 - v_t v_x \xi_v^1 - v_y \xi_x^3 - v_x v_y \xi_v^3. \quad (3.9)$$

Using eqns(3.6c) and (3.7c) we have that

$$\zeta_y = \eta_y + v_y(\eta_v - \xi_y^3) - v_y^2 \xi_v^3 - v_t \xi_y^1 - v_t v_y \xi_v^1 - v_x \xi_y^2 - v_x v_y \xi_v^2. \quad (3.10)$$

Then from eqns (3.6b), (3.9), and (3.7d) we have that

$$\begin{aligned} \zeta_{xx} &= \eta_{xx} + v_x \eta_{vx} + v_{xx} \eta_v - v_{xx} \xi_x^2 + v_x \eta_{vx} + v_x^2 \eta_{vv} \\ &\quad - v_x \xi_{xx}^2 - v_x^2 \xi_{vx}^2 - v_x^2 \xi_{vv}^2 - v_x^3 \xi_{vv}^2 - 2v_x v_{xx} \xi_v^2 - v_{tx} \xi_x^1 \\ &\quad - v_t \xi_{xx}^1 - v_t v_x \xi_{vx}^1 - v_t v_x \xi_{vv}^1 - v_t v_x^2 \xi_{vv}^1 - v_t v_{xx} \xi_v^1 - v_x v_{tx} \xi_v^1 \\ &\quad - v_x v_y \xi_{vx}^3 - v_x^2 v_y \xi_{vv}^3 - v_x v_{xy} \xi_v^3 - v_y v_{xx} \xi_v^3 - v_y \xi_{xx}^3 - v_x v_y \xi_{vx}^3 \\ &\quad - v_{xy} \xi_x^3 - v_{tx} \xi_x^1 - v_x v_{tx} \xi_v^1 - v_{xx} \xi_x^2 - v_x v_{xx} \xi_v^2 - v_{xy} \xi_x^3 - v_x v_{xy} \xi_v^3, \end{aligned} \quad (3.11)$$

which simplifies to

$$\begin{aligned}
\zeta_{xx} &= \eta_{xx} + 2v_x\eta_{vx} + v_{xx}\eta_v - 2v_{xx}\xi_x^2 + v_x^2\eta_{vv} - v_x\xi_{xx}^2 \\
&\quad - 2v_x^2\xi_{vx}^2 - v_x^3\xi_{vv}^2 - 3v_xv_{xx}\xi_v^2 - 2v_{tx}\xi_x^1 - v_t\xi_{xx}^1 \\
&\quad - 2v_tv_x\xi_{vx}^1 - v_tv_x^2\xi_{vv}^1 - v_tv_{xx}\xi_v^1 - 2v_xv_{tx}\xi_v^1 - 2v_xv_y\xi_{vx}^3 \\
&\quad - v_x^2v_y\xi_{vv}^3 - 2v_xv_{xy}\xi_v^3 - v_yv_{xx}\xi_v^3 - v_y\xi_{xx}^3 - 2v_{xy}\xi_x^3.
\end{aligned} \tag{3.12}$$

Substituting eqns (3.8), (3.9), (3.10), and (3.12) into eqn (3.5), we have that

$$\begin{aligned}
&2\xi^2(rv_x + \sigma^2xv_{xx} + v_y) - 2r\eta + 2(\eta_t + v_t(\eta_v - \xi_t^1) - v_t^2\xi_v^1 - v_x\xi_t^2 \\
&\quad - v_tv_x\xi_v^2 - v_y\xi_t^3 - v_tv_y\xi_v^3) + 2rx(\eta_x + v_x(\eta_v - \xi_x^2) - v_x^2\xi_v^2 - v_t\xi_x^1 \\
&\quad - v_tv_x\xi_v^1 - v_y\xi_x^3 - v_xv_y\xi_v^3) + 2x(\eta_y + v_y(\eta_v - \xi_y^3) - v_y^2\xi_v^3 - v_t\xi_y^1 \\
&\quad - v_tv_y\xi_v^1 - v_x\xi_y^2 - v_xv_y\xi_v^2) + \sigma^2x^2(\eta_{xx} + 2v_x\eta_{vx} + v_{xx}\eta_v - 2v_{xx}\xi_x^2 \\
&\quad + v_x^2\eta_{vv} - v_x\xi_{xx}^2 - 2v_x^2\xi_{vx}^2 - v_x^3\xi_{vv}^2 - 3v_xv_{xx}\xi_v^2 - 2v_{tx}\xi_x^1 - v_t\xi_{xx}^1 \\
&\quad - 2v_tv_x\xi_{vx}^1 - v_tv_x^2\xi_{vv}^1 - v_tv_{xx}\xi_v^1 - 2v_xv_{tx}\xi_v^1 - 2v_xv_y\xi_{vx}^3 - v_x^2v_y\xi_{vv}^3 \\
&\quad - 2v_xv_{xy}\xi_v^3 - v_yv_{xx}\xi_v^3 - v_y\xi_{xx}^3 - 2v_{xy}\xi_x^3) = 0.
\end{aligned} \tag{3.13}$$

Now, from eqn (1.3) we can have that

$$v_t = rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y. \tag{3.14}$$

When considering eqn (3.14), eqn (3.13) becomes

$$\begin{aligned}
& 2\xi^2rv_x + 2\xi^2\sigma^2xv_{xx} + 2\xi^2v_y - 2r\eta + 2\eta_t + 2(rv - rxv_x \\
& - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)(\eta_v - \xi_t^1) - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
& - xv_y)^2\xi_v^1 - 2v_x\xi_t^2 - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi_v^2 \\
& - 2v_y\xi_t^3 - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_y\xi_v^3 + 2rx\eta_x \\
& + 2rxv_x(\eta_v - \xi_x^2) - 2rxv_x^2\xi_v^2 - 2rx(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
& - xv_y)\xi_x^1 - 2rx(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi_v^1 - 2rxv_y\xi_x^3 \\
& - 2rxv_xv_y\xi_v^3 + 2x\eta_y + 2xv_y(\eta_v - \xi_y^3) - 2xv_y^2\xi_v^3 - 2x(rv - rxv_x \\
& - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)\xi_y^1 - 2x(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_y\xi_v^1 \\
& - 2xv_x\xi_y^2 - 2xv_xv_y\xi_v^2 + \sigma^2x^2\eta_{xx} + 2\sigma^2x^2v_x\eta_{vx} + \sigma^2x^2v_{xx}\eta_v \\
& - 2\sigma^2x^2v_{xx}\xi_x^2 + \sigma^2x^2v_x^2\eta_{vv} - \sigma^2x^2v_x\xi_{xx}^2 - 2\sigma^2x^2v_x^2\xi_{vx}^2 - \sigma^2x^2v_x^3\xi_{vv}^2 \\
& - 3\sigma^2x^2v_xv_{xx}\xi_v^2 - 2\sigma^2x^2v_{tx}\xi_x^1 - \sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
& - xv_y)\xi_{xx}^1 - 2\sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi_{vx}^1 \\
& - \sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x^2\xi_{vv}^1 - \sigma^2x^2(rv - rxv_x \\
& - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_{xx}\xi_v^1 - 2\sigma^2x^2v_xv_{tx}\xi_v^1 - 2\sigma^2x^2v_xv_y\xi_{vx}^3 \\
& - \sigma^2x^2v_x^2v_y\xi_{vv}^3 - 2\sigma^2x^2v_xv_{xy}\xi_v^3 - \sigma^2x^2v_yv_{xx}\xi_v^3 - \sigma^2x^2v_y\xi_{xx}^3 \\
& - 2\sigma^2x^2v_{xy}\xi_x^3 = 0,
\end{aligned} \tag{3.15}$$

which simplifies to

$$\begin{aligned}
& 2\xi^2rv_x + 2\xi^2\sigma^2xv_{xx} + 2\xi^2v_y - 2r\eta + 2\eta_t + 2rv\eta_v - 2rxv_x\eta_v \\
& - \sigma^2x^2v_{xx}\eta_v - 2xv_y\eta_v - 2rv\xi_t^1 + 2rxv_x\xi_t^1 + \sigma^2x^2v_{xx}\xi_t^1 + 2xv_y\xi_t^1 \\
& - 2r^2v^2\xi_v^1 + 4vr^2xv_x\xi_v^1 + 4xvrvv_y\xi_v^1 - 2r^2x^2v_x^2\xi_v^1 - 4rx^2v_xv_y\xi_v^1 \\
& - 2x^2v_y^2\xi_v^1 - 2rx^3\sigma^2v_xv_{xx}\xi_v^1 - 2\sigma^2x^3v_yv_{xx}\xi_v^1 - \frac{1}{2}\sigma^4x^4v_{xx}^2\xi_v^1 \\
& + 2rv\sigma^2x^2v_{xx}\xi_v^1 - 2v_x\xi_t^2 - 2rvv_x\xi_v^2 + 2rxv_x^2\xi_v^2 + \sigma^2x^2v_{xx}v_x\xi_v^2 \\
& + 2xv_yv_x\xi_v^2 - 2v_y\xi_t^3 - 2rvv_y\xi_v^3 + 2rxv_xv_y\xi_v^3 + \sigma^2x^2v_{xx}v_y\xi_v^3 \\
& + 2xv_y^2\xi_v^3 + 2rx\eta_x + 2rxv_x\eta_v - 2rxv_x\xi_x^2 - 2rxv_x^2\xi_v^2 - 2r^2xv\xi_x^1 \\
& + 2r^2x^2v_x\xi_x^1 + r\sigma^2x^3v_{xx}\xi_x^1 + 2rx^2v_y\xi_x^1 - 2r^2xv_{xx}\xi_v^1 + 2r^2x^2v_x^2\xi_v^1 \\
& + r\sigma^2x^3v_{xx}v_x\xi_v^1 + 2rx^2v_yv_x\xi_v^1 - 2rxv_y\xi_x^3 - 2rxv_xv_y\xi_v^3 + 2x\eta_y \\
& + 2xv_y\eta_v - 2xv_y\xi_y^3 - 2xv_y^2\xi_v^3 - 2xrv\xi_y^1 + 2x^2rv_x\xi_y^1 + \sigma^2x^3v_{xx}\xi_y^1 \\
& + 2x^2v_y\xi_y^1 - 2xrvv_y\xi_v^1 + 2x^2rv_xv_y\xi_v^1 + \sigma^2x^3v_{xx}v_y\xi_v^1 + 2x^2v_y^2\xi_v^1 \\
& - 2xv_x\xi_y^2 - 2xv_xv_y\xi_v^2 + \sigma^2x^2\eta_{xx} + 2\sigma^2x^2v_x\eta_{vx} + \sigma^2x^2v_{xx}\eta_v \\
& - 2\sigma^2x^2v_{xx}\xi_x^2 + \sigma^2x^2v_x^2\eta_{vv} - \sigma^2x^2v_x\xi_{xx}^2 - 2\sigma^2x^2v_x^2\xi_{vx}^2 - \sigma^2x^2v_x^3\xi_{vv}^2 \\
& - 3\sigma^2x^2v_xv_{xx}\xi_v^2 - 2\sigma^2x^2v_{tx}\xi_x^1 - \sigma^2x^2rv\xi_{xx}^1 + \sigma^2x^3rv_x\xi_{xx}^1 \\
& + \frac{1}{2}\sigma^4x^4v_{xx}\xi_{xx}^1 + \sigma^2x^3v_y\xi_{xx}^1 - 2\sigma^2x^2rvv_x\xi_{vx}^1 + 2\sigma^2x^3rv_x^2\xi_{vx}^1 \\
& + \sigma^4x^4v_{xx}v_x\xi_{vv}^1 + 2\sigma^2x^3v_yv_x\xi_{vv}^1 - \sigma^2x^2rvv_x^2\xi_{vv}^1 + \sigma^2x^3rv_x^3\xi_{vv}^1 \\
& + \frac{1}{2}\sigma^4x^4v_{xx}v_x^2\xi_{vv}^1 + \sigma^2x^3v_yv_x^2\xi_{vv}^1 - \sigma^2x^2rvv_{xx}\xi_v^1 + \sigma^2x^3rv_xv_{xx}\xi_v^1 \\
& + \frac{1}{2}\sigma^4x^4v_{xx}^2\xi_v^1 + \sigma^2x^3xv_yv_{xx}\xi_v^1 - 2\sigma^2x^2v_xv_{tx}\xi_v^1 - 2\sigma^2x^2v_xv_y\xi_{vx}^3 \\
& - \sigma^2x^2v_x^2v_y\xi_{vv}^3 - 2\sigma^2x^2v_xv_{xy}\xi_v^3 - \sigma^2x^2v_yv_{xx}\xi_v^3 - \sigma^2x^2v_y\xi_{xx}^3 \\
& - 2\sigma^2x^2v_{xy}\xi_x^3 = 0.
\end{aligned} \tag{3.16}$$

Now, separating eqn (3.16) w.r.t powers and products of v we get the following determining equations:

$$v_{tx} : \xi_x^1 = 0, \quad (3.17a)$$

$$v_x v_{xy} : \xi_v^3 = 0, \quad (3.17b)$$

$$v_{xy} : \xi_x^3 = 0, \quad (3.17c)$$

$$v_x v_{tx} : \xi_v^1 = 0, \quad (3.17d)$$

$$v_{xx} v_x^2 : \xi_{vv}^1 = 0, \quad (3.17e)$$

$$v_y v_x^2 : x \xi_{vv}^1 - \xi_{vv}^3 = 0, \quad (3.17f)$$

$$v_x v_y : -2\xi_{vx}^3 + 2x \xi_{vx}^1 = 0, \quad (3.17g)$$

$$v_x^3 : -\xi_{vv}^2 + 2rx \xi_{vv}^1 = 0, \quad (3.17h)$$

$$v_x v_{xx} : -\xi_v^2 + \sigma^2 x^2 \xi_{vx}^1 = 0, \quad (3.17i)$$

$$v_x^2 : \eta_{vv} - 2\xi_{vx}^2 - 2rx \xi_{vx}^1 - rv \xi_{vv}^1 = 0, \quad (3.17j)$$

$$v_{xx} : 4\xi^2 + 2x \xi_t^1 + 2rvx \xi_v^1 + 2rx^2 \xi_x^1 + 2x^2 \xi_y^1 - 4x \xi_x^2 + \sigma^2 x^3 \xi_{xx}^1 = 0, \quad (3.17k)$$

$$\begin{aligned} v_y : & 2\xi^2 + 2x \xi_t^1 + 2rvx \xi_v^1 - 2\xi_t^3 - 2rv \xi_v^3 + 2rx^2 \xi_x^1 - 2rx \xi_x^3 \\ & + \sigma^2 x^3 \xi_{xx}^1 - \sigma^2 x^2 \xi_{xx}^3 - 2x \xi_y^3 + 2x^2 \xi_y^1 = 0, \end{aligned} \quad (3.17l)$$

$$\begin{aligned} v_x : & 2\xi^2 r + 2rx \xi_t^1 - 2r^2 xv \xi_v^1 - 2\xi_t^2 - 2rv \xi_v^2 - 2rx \xi_x^2 + 2r^2 x^2 \xi_x^1 \\ & + 2rx^2 \xi_y^1 - 2x \xi_y^2 + 2\sigma^2 x^2 \eta_{vx} - r\sigma^2 x^3 \xi_{xx}^1 - 2rv\sigma^2 x^2 \xi_{vx}^1 = 0, \end{aligned} \quad (3.17m)$$

$$\begin{aligned} \text{Constant-terms: } & -2r\eta + 2\eta_t + 2rv\eta_v - 2rv \xi_t^1 + 2rx \eta_x - 2r^2 xv \xi_x^1 \\ & - 2rvx \xi_y^1 + \sigma^2 x^2 \eta_{xx} - rv\sigma^2 x^2 \xi_{xx}^1 - 2r^2 v^2 \xi_v^1 + 2x \eta_y = 0. \end{aligned} \quad (3.17n)$$

3.2 Solving the Determining Equations

From eqn (3.17a), we have that

$$\xi^1 = \xi^1(t, y, v). \quad (3.18)$$

From eqn (3.17d), we have that

$$\xi^1 = \xi^1(t, x, y). \quad (3.19)$$

Now, from eqns (3.18) and (3.19) we have that

$$\xi^1 = \xi^1(t, y). \quad (3.20)$$

When considering eqn (3.17b), we have that

$$\xi^3 = \xi^3(t, x, y), \quad (3.21)$$

and from eqn (3.17c) we have that

$$\xi^3 = \xi^3(t, y, v). \quad (3.22)$$

Using eqns (3.22) and (3.21) we have that

$$\xi^3 = \xi^3(t, y). \quad (3.23)$$

From eqns (3.17e) and (3.17f), we have that

$$\xi_{vv}^3 = 0, \quad (3.24)$$

which is identically satisfied by eqn (3.23). By eqns (3.17g), (3.20), and (3.23), eqn (3.17g) is satisfied. Taking eqns (3.17d), (3.20), and (3.17i), we have that

$$\xi^2 = \xi^2(t, x, y). \quad (3.25)$$

Again, when taking eqns (3.17e) and (3.17h), we have that

$$\xi_{vv}^2 = 0, \quad (3.26)$$

which is identically satisfied by eqn (3.25). Considering eqns (3.17j), (3.17d), (3.17e), (3.20), and (3.25), we have that

$$\begin{aligned} \eta_{vv} &= 0, \\ \Rightarrow \eta &= a(t, x, y)v + b(t, x, y). \end{aligned} \quad (3.27)$$

Equations (3.17l), (3.20), and (3.23) yield

$$x\xi_t^1 - \xi_t^3 + x^2\xi_y^1 - x\xi_y^3 + \xi^2 = 0, \quad (3.28)$$

which when differentiated w.r.t x thrice gives

$$\xi^2(t, x, y) = c(t, y) + xd(t, y) + x^2e(t, y). \quad (3.29)$$

Substituting eqn (3.29) into eqn (3.28) and separating w.r.t powers of x we have that

$$\begin{aligned} c(t, y) &= \xi_t^3, \\ d(t, y) &= \xi_y^3 - \xi_t^1, \\ e(t, y) &= -\xi_y^1. \end{aligned} \quad (3.30)$$

Then, from eqns (3.29) and (3.30) we have that

$$\xi^2(t, x, y) = \xi_t^3 + x\xi_y^3 - x\xi_t^1 - x^2\xi_y^1. \quad (3.31)$$

Now, from eqns (3.17k), (3.17a), (3.20), (3.27) we have that

$$2\xi^2 + x^2\xi_y^1 + x\xi_t^1 - 2x\xi_x^2 = 0. \quad (3.32)$$

Using eqns (3.31) and (3.32), we have that

$$\begin{aligned} & 2(\xi_t^3 + x\xi_y^3 - x\xi_t^1 - x^2\xi_y^1) + x^2\xi_y^1 + x\xi_t^1 - 2x(-2x\xi_y^1 + \xi_y^3 - \xi_t^1) = 0 \\ \Rightarrow & 3x^2\xi_y^1 + x\xi_t^1 + 2\xi_t^3 = 0. \end{aligned} \quad (3.33)$$

Differentiating eqn (3.33) w.r.t x twice we have that

$$\begin{aligned} & 18\xi_y^1 = 0, \\ \Rightarrow & \xi^1 = \xi^1(t). \end{aligned} \quad (3.34)$$

And thus, eqns (3.33) and (3.34) yield

$$x\xi_t^1 + 2\xi_t^3 = 0, \quad (3.35)$$

which when differentiated w.r.t x gives

$$\begin{aligned} & \xi_t^1 = 0, \\ \Rightarrow & \xi^1(t, x, y, v) = K_1. \end{aligned} \quad (3.36)$$

Thus, from eqns (3.35) and (3.36), we have that

$$\begin{aligned}\xi_t^3 &= 0, \\ \Rightarrow \xi^3 &= \xi^3(y).\end{aligned}\tag{3.37}$$

When considering eqns (3.17m), (3.20), and (3.23), we have that

$$2rx^2\xi_y^1 - 2x\xi_y^2 - 2rx\xi_x^2 + 2\sigma^2x^2\eta_{vx} - \sigma^2x^2\xi_{xx}^2 + 2rx\xi_t^1 - 2\xi_t^1 = 0.\tag{3.38}$$

Taking into consideration eqns (3.31) (3.36), and (3.37), eqn (3.38) becomes

$$-x\xi_{yy}^3 - r\xi_y^3 + x\sigma^2a_x = 0.\tag{3.39}$$

Differentiating eqn (3.39) w.r.t x gives and solving the output yields

$$a(t, x, y) = f(t, y) + \frac{x\xi_{yy}^3}{\sigma^2}.\tag{3.40}$$

When substituting eqn (3.40) into eqn (3.27) we have that

$$\eta = \left(f(t, y) + \frac{x\xi_{yy}^3}{\sigma^2} \right) v + b(t, x, y).\tag{3.41}$$

From eqns (3.17n) and (3.20), we have that

$$-2r\eta + 2rv\eta_v + 2\eta_t + 2rx\eta_x + x^2\sigma^2\eta_{xx} + 2x\eta_y - 2rv(x\xi_y^1 + \xi_t^1) = 0,\tag{3.42}$$

which when considering eqns (3.36) and (3.27), becomes

$$\begin{aligned}vx(a_y) + x(b_y) + rvx(a_x) + rx(b_x) + \frac{1}{2}vx^2\sigma^2(a_{xx}) \\ + \frac{1}{2}x^2\sigma^2(b_{xx}) + v(a_t) + b_t - rb = 0.\end{aligned}\tag{3.43}$$

When considering eqn (3.40), eqn (3.43) becomes

$$\frac{rvx\xi_{yy}^3}{\sigma^2} + \frac{vx^2\xi_{yyy}^3}{\sigma^2} + vx(f_y) + v(f_t) + x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \quad (3.44)$$

Differentiating eqn (3.44) w.r.t v gives

$$f_t + xf_y + \frac{rx\xi_{yy}^3}{\sigma^2} + \frac{x^2\xi_{yyy}^3}{\sigma^2} = 0. \quad (3.45)$$

Again, differentiating eqn (3.45) w.r.t x twice gives

$$\frac{2\xi_{yyy}^3}{\sigma^2} = 0, \quad (3.46)$$

which when solved yields

$$\xi^3(t, x, y, v) = K_2 + yK_3 + y^2K_4. \quad (3.47)$$

Now, when considering eqns (3.44) and (3.47) we have that

$$\frac{2rvxK_4}{\sigma^2} + vx(f_y) + v(f_t) + x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0, \quad (3.48)$$

which when differentiated w.r.t v , then w.r.t x gives

$$\frac{2rK_4}{\sigma^2} + f_y = 0. \quad (3.49)$$

Solving for f on eqn (3.49) yields

$$f = -\frac{2ryK_4}{\sigma^2} + g(t). \quad (3.50)$$

Now, from eqns (3.48) and (3.50) we have that

$$v(g_t) + x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \quad (3.51)$$

Differentiating eqn (3.51) w.r.t v gives

$$\begin{aligned} g_t &= 0, \\ \Rightarrow g &= K_5. \end{aligned} \quad (3.52)$$

Taking into consideration eqn (3.52), eqn (3.51) becomes

$$x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \quad (3.53)$$

Now, when considering eqns (3.31), (3.36), and (3.47) we have that

$$\xi^2(t, x, y, v) = xK_3 + 2xyK_4. \quad (3.54)$$

Again, when considering eqns (3.41), (3.47), (3.50), and (3.52) we have that

$$\eta(t, x, y, v) = v \left(\frac{2xK_4 - 2ryK_4}{\sigma^2} + K_5 \right) + b(t, x, y). \quad (3.55)$$

Upon substituting eqns (3.36), (3.47), (3.54), and (3.55) into (3.1) gives

$$\begin{aligned} X &= K_1\partial_t + (xK_3 + 2xyK_4)\partial_x + (K_2 + yK_3 + y^2K_4)\partial_y \\ &+ \left(v \left(\frac{2xK_4 - 2ryK_4}{\sigma^2} + K_5 \right) + b(t, x, y) \right) \partial_v, \end{aligned} \quad (3.56)$$

where K_1, K_2, K_3, K_4 and K_5 are constants, and $b(t, x, y)$ is some function satisfying (1.3).

3.3 The Infinitesimal Generators

The resulting symmetries for eqn (1.3) are

$$X_1 = \partial_t, \tag{3.57a}$$

$$X_2 = \partial_y, \tag{3.57b}$$

$$X_3 = x\partial_x + y\partial_y, \tag{3.57c}$$

$$X_4 = 2xy\partial_x + y^2\partial_y + v\left(\frac{2x - 2ry}{\sigma^2}\right)\partial_v, \tag{3.57d}$$

$$X_5 = v\partial_v, \tag{3.57e}$$

$$X_b = b(t, x, y)\partial_v, \tag{3.57f}$$

where $b(t, x, y)$ satisfies eqn (1.3).

CHAPTER 4

OPTIMAL SYSTEM, SYMMETRY REDUCTION AND GROUP INVARIANT SOLUTIONS

In this chapter, we construct an optimal system of the arithmetic Asian options PDE as well as performing the symmetry reductions and finding the invariant solutions.

4.1 Optimal System of One-dimensional Sub-algebras

There are several methods for constructing optimal systems of Lie algebras, including Ovsianikov's approach [85]. This dissertation adopts Olver's method [86, 87, 88, 89] for its systematic algorithmic structure and demonstrated success in simplifying complex PDEs.

4.1.1 Lie Brackets

The notes on finding the Lie brackets (A commutator of symmetries group) can be found in Section 2.4. The resulting Lie brackets table is as shown in Table 4.1.

Table 4.1: Lie Brackets Table

$\nearrow [X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	0	0	0	0
X_2	0	0	X_2	$2X_3 - \frac{2r}{\sigma^2}X_5$	0
X_3	0	$-X_2$	0	X_4	0
X_4	0	$-(2X_3 - \frac{2r}{\sigma^2}X_5)$	$-X_4$	0	0
X_5	0	0	0	0	0

4.1.2 Adjoint Representation

The adjoint representation describes how each infinitesimal generator transforms other generators via the Lie bracket. Specifically, for a generator X_i , the adjoint action on X_j is expressed as ¹:

$$\text{Ad}(e^{\epsilon X_i})X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \dots \quad (4.1)$$

With the help of Table 4.1 and eqn (4.1), the resulting adjoint representation table is as follows

Table 4.2: Adjoint Representation Table

$\text{Ad}[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	X_3	X_4	X_5
X_2	X_1	X_2	$X_3 - \epsilon X_2$	$X_4 - \epsilon(2X_3 - \frac{2r}{\sigma^2}X_5) + \epsilon^2 X_2$	X_5
X_3	X_1	$e^\epsilon X_2$	X_3	$e^{-\epsilon} X_4$	X_5
X_4	X_1	$X_2 + \epsilon 2X_3 - \epsilon \frac{2r}{\sigma^2} X_5 + \epsilon^2 X_4$	$X_3 + \epsilon X_4$	X_4	X_5
X_5	X_1	X_2	X_3	X_4	X_5

¹see [90, 91].

4.1.3 Optimal System

To build an optimal system of invariant solutions, consider the linear combinations of the symmetries

$$X = \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3 + \gamma_4 X_4 + \gamma_5 X_5, \quad (4.2)$$

where $\gamma_1 \dots \gamma_5$ are arbitrary constants. To find the optimal system for eqn (1.3), the coefficients in eqn (4.2) should be simplified as much as possible with the help of Table 4.2².

Case 1: Suppose $\gamma_1 = 1$

Equation (4.2) becomes

$$X = X_1 + \gamma_2 X_2 + \gamma_3 X_3 + \gamma_4 X_4 + \gamma_5 X_5. \quad (4.3)$$

From the adjoint table, $\text{Ad}(e^{\epsilon X_1})X_j = X_j$ for $j = [2,5]$, where $j \in \mathbb{R}$, thus no further simplification is possible. Applying $\text{Ad}(e^{\epsilon X_2})$ on eqn (4.3) eliminates X_2 , as shown below

$$\begin{aligned} X' &= \text{Ad}(e^{\epsilon X_2})X \\ &= \text{Ad}(e^{\epsilon X_2})X_1 + \gamma_2 \text{Ad}(e^{\epsilon X_2})X_2 + \gamma_3 \text{Ad}(e^{\epsilon X_2})X_3 \\ &\quad + \gamma_4 \text{Ad}(e^{\epsilon X_2})X_4 + \gamma_5 \text{Ad}(e^{\epsilon X_2})X_5 \\ &= X_1 + \gamma_2 X_2 + \gamma_3 (X_3 - \epsilon X_2) + \gamma_4 \left(X_4 - \epsilon \left(2X_3 - \frac{2r}{\sigma^2} X_5 \right) + \epsilon^2 X_2 \right) + \gamma_5 X_5 \\ &= X_1 + \gamma_2 X_2 - \gamma_3 \epsilon X_2 + \epsilon^2 X_2 + (\gamma_3 - 2\gamma_4 \epsilon) X_3 + \gamma_4 X_4 + \left(\frac{2\gamma_4 \epsilon r}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.4)$$

²The higher-order terms in the adjoint representation, such as $\epsilon^2 X_2$ and $\epsilon^2 X_4$, which arise from the Baker–Campbell–Hausdorff expansion (e.g., $\frac{\epsilon^2}{2} [X_i, [X_i, X_j]]$ and higher), do not affect the construction of the optimal system. When choosing the parameter ϵ to eliminate a generator, these quadratic contributions either vanish identically or become proportional to generators that have already been removed. Consequently, the linear-order terms are sufficient for the simplification and normalization steps, and the final set of distinct generators in the optimal system remains unchanged.

Now, in order to eliminate X_2 , let $\epsilon = \frac{\gamma_2}{\gamma_3}$. This result into

$$\begin{aligned} X' &= X_1 + \gamma_2 X_2 - \gamma_3 \frac{\gamma_2}{\gamma_3} X_2 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) X_3 + \gamma_4 X_4 + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5 \\ &= X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) X_3 + \gamma_4 X_4 + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.5)$$

To eliminate X_4 , we apply $Ad(e^{\epsilon X_4})$ on X' as follows,

$$\begin{aligned} X'' &= Ad(e^{\epsilon X_4})X' \\ &= Ad(e^{\epsilon X_4})X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) Ad(e^{\epsilon X_4})X_3 + \gamma_4 Ad(e^{\epsilon X_4})X_4 \\ &\quad + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) Ad(e^{\epsilon X_4})X_5 \\ &= X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) (X_3 + \epsilon X_4) + \gamma_4 X_4 + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5 \\ &= X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) X_3 + \left(\left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) \epsilon + \gamma_4 \right) X_4 + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.6)$$

Let $\epsilon = \frac{-\gamma_4}{\left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right)}$ in order to eliminate X_4 . We thus have

$$\begin{aligned} X'' &= X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) X_3 + \left(\left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) \frac{-\gamma_4}{\left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right)} + \gamma_4 \right) X_4 \\ &\quad + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5 \\ &= X_1 + \left(\gamma_3 - 2\gamma_4 \frac{\gamma_2}{\gamma_3} \right) X_3 + \left(\frac{2\gamma_4 \frac{\gamma_2}{\gamma_3} r}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.7)$$

Taking the coefficients of X_3 and X_5 on eqn (4.7) to be ± 1 simplifies eqn (4.3) to either of the following

$$X_1 + X_3 + X_5, \quad X_1 - X_3 + X_5, \quad X_1 - X_3 - X_5, \quad X_1 + X_3 - X_5. \quad (4.8)$$

No further simplifications can be made.

Case 2: Suppose $\gamma_1 = 0$, and let $\gamma_2 = 1$

In this case, the generator starts as:

$$X = X_2 + \gamma_3 X_3 + \gamma_4 X_4 + \gamma_5 X_5 \quad (4.9)$$

Acting on eqn (4.9) by $Ad(e^{\epsilon X_4})$ eliminates X_4 as thus

$$\begin{aligned} X' &= Ad(e^{\epsilon X_4})X \\ &= Ad(e^{\epsilon X_4})X_2 + \gamma_3 Ad(e^{\epsilon X_4})X_3 + \gamma_4 Ad(e^{\epsilon X_4})X_4 + \gamma_5 Ad(e^{\epsilon X_4})X_5 \\ &= X_2 + \epsilon \left(2X_3 - \frac{2r}{\sigma^2} X_5 \right) + \gamma_3 (X_1 + \epsilon X_4) + \gamma_4 X_4 + \gamma_5 X_5 \\ &= X_2 + (2\epsilon + \gamma_3)X_3 + (\gamma_3\epsilon + \gamma_4)X_4 + \left(-\frac{2r\epsilon}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.10)$$

Let $\epsilon = \frac{-\gamma_4}{\gamma_3}$, we then have

$$\begin{aligned} X' &= X_2 + \left(2\frac{-\gamma_4}{\gamma_3} + \gamma_3 \right) X_3 + \left(\gamma_3 \frac{-\gamma_4}{\gamma_3} + \gamma_4 \right) X_4 + \left(-\frac{2r\frac{-\gamma_4}{\gamma_3}}{\sigma^2} + \gamma_5 \right) X_5 \\ &= X_2 + \left(\frac{-2\gamma_4}{\gamma_3} + \gamma_3 \right) X_3 + \left(\frac{2r\gamma_4}{\sigma^2} + \gamma_5 \right) X_5. \end{aligned} \quad (4.11)$$

Taking the coefficients of X_3 and X_5 on eqn (4.11) to be ± 1 simplifies eqn (4.9) to either of the following

$$X_2 + X_3 + X_5, \quad X_2 - X_3 + X_5, \quad X_2 - X_3 - X_5, \quad X_2 + X_3 - X_5. \quad (4.12)$$

Case 3: Suppose $\gamma_1 = \gamma_2 = 0$, and let $\gamma_3 = 1$

Here, the initial generator is:

$$X = X_3 + \gamma_4 X_4 + \gamma_5 X_5.$$

We apply $Ad(e^{\epsilon X_4})$ to eliminate X_4 as follows

$$\begin{aligned} X' &= Ad(e^{\epsilon X_4})X \\ &= Ad(e^{\epsilon X_4})X_3 + \gamma_4 Ad(e^{\epsilon X_4})X_4 + \gamma_5 Ad(e^{\epsilon X_4})X_5 \\ &= X_3 + \epsilon X_4 + \gamma_4 X_4 + \gamma_5 X_5. \end{aligned} \tag{4.13}$$

Let $\epsilon = -\gamma_4$, then we have that

$$\begin{aligned} X' &= X_3 - \gamma_4 X_4 + \gamma_4 X_4 + \gamma_5 X_5 \\ &= X_3 + a X_5, \end{aligned} \tag{4.14}$$

where a is any real constants. No further simplifications can be made.

Case 4: Suppose $\gamma_1 = \gamma_2 = \gamma_3 = 0$, and let $\gamma_4 = 1$

Here we simplify the generator to:

$$X = X_4 + \gamma_5 X_5. \tag{4.15}$$

The adjoint actions between X_4 and X_5 do not allow further simplification (since their commutators are zero). Thus we have

$$X = X_4 + a X_5, \tag{4.16}$$

where a is a real constant.

Case 5: Suppose $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \mathbf{0}$, and let $\gamma_5 = \mathbf{1}$

Equation (4.2) can be simplified to:

$$X = X_5. \quad (4.17)$$

There are no further simplifications that can be made.

Thus, the resulting optimal system for eqn (1.3) is given by the following set of distinct combinations:

$$\left\{ \begin{array}{cccc} X_1 + X_3 + X_5, & X_1 - X_3 + X_5, & X_1 - X_3 - X_5, & X_1 + X_3 - X_5, \\ X_2 + X_3 + X_5, & X_2 - X_3 + X_5, & X_2 - X_3 - X_5, & X_2 + X_3 - X_5, \\ & & X_3 + aX_5, & X_4 + aX_5, & X_5 \end{array} \right\}, \quad (4.18)$$

where $a \in \mathbb{R}$.

4.2 Symmetry Reduction and Group Invariant Solutions

Recall that the infinitesimal generator for eqn (1.3) w.r.t the generator (2.5) is given by

$$X = \xi^1(t, x, y, v) \partial_t + \xi^2(t, x, y, v) \partial_x + \xi^3(t, x, y, v) \partial_y + \eta(t, x, y, v) \partial_v. \quad (4.19)$$

Now, for the given infinitesimal generator (4.19), the characteristic equations are derived as follows:

$$\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{dy}{\xi^3} = \frac{dv}{\eta}. \quad (4.20)$$

These equations describe the invariant surfaces for the generator and can be used to reduce the number of independent variables in the model PDE (1.3), satisfying eqn (2.20). From section 4.1.3, case 1 and case 2 gave four possible combinations as optimal systems to the

model PDE.

4.2.1 Case 1

When considering $X_1 + X_3 + X_5$ gives the following characteristic equations

$$\frac{dt}{1} = \frac{dx}{x} = \frac{dy}{y} = \frac{dv}{v}. \quad (4.21)$$

From the first and third ratios we have that

$$\begin{aligned} \frac{dt}{1} &= \frac{dy}{y}, \\ \Rightarrow z_1 &= ye^{-t}, \end{aligned} \quad (4.22)$$

where z_1 is some new constant of integration. From the first and second ratios we get

$$\begin{aligned} \frac{dt}{1} &= \frac{dx}{x}, \\ \Rightarrow z_2 &= xe^{-t}, \end{aligned} \quad (4.23)$$

where z_2 is some new constant of integration. Taking the first and last ratios gives

$$\begin{aligned} \frac{dt}{1} &= \frac{dv}{v}, \\ \Rightarrow v &= e^t C_2, \end{aligned} \quad (4.24)$$

where C_2 is a constant of integration. The invariants are as follows

$$z_1 = ye^{-t}, \quad z_2 = xe^{-t}, \quad v(t, x, y) = e^t F(z_1, z_2), \quad (4.25)$$

for some function F that depends on the new variables z_1 and z_2 . Upon substituting the invariants (4.25) into eqn (1.3) gives

$$(1 - r)F + (z_2 - z_1)F_{z_1} + (rz_2 - z_2)F_{z_2} + \frac{1}{2}z_2^2\sigma^2F_{z_2z_2} = 0. \quad (4.26)$$

Equation (4.26) has to be simplified further into an ODE, and we do so by finding its symmetries which are given as

$$Y_1 = z_1\partial_{z_1} + z_2\partial_{z_2}, \quad Y_2 = F\partial_F, \quad Y_b = b(z_1, z_2). \quad (4.27)$$

These symmetries were found using mathematica software [92] in conjunction with the sym package [93]³. Next we consider a reduction of eqn (4.26) using the linear combinations of finite symmetries in (4.27).

Invariance Under Y_1

The resulting characteristic equations are

$$\frac{dz_1}{z_1} = \frac{dz_2}{z_2} = \frac{dF}{0}, \quad (4.28)$$

which yield the invariants

$$w = \frac{z_1}{z_2}, \quad F(z_1, z_2) = g(w). \quad (4.29)$$

Substituting the invariants (4.29) into eqn (4.26) we have an ODE

$$(1 - r)g + (1 - rw + w\sigma^2)g' + \frac{1}{2}w^2\sigma^2g'' = 0. \quad (4.30)$$

³These symmetries are the same for the other reduced PDEs on this case under other linear combinations.

Invariance Under $Y_1 + Y_2$

The characteristic equations are

$$\frac{dz_1}{z_1} = \frac{dz_2}{z_2} = \frac{dF}{F}, \quad (4.31)$$

and they give the invariants

$$w = \frac{z_1}{z_2}, \quad F(z_1, z_2) = z_1 H(w). \quad (4.32)$$

Substituting the invariants (4.32) into eqn (4.26) gives the same ODE as in eqn (4.30). Table 4.3 shows reductions under other combinations for this case, whereby $X_1 + X_3 + X_5$ and $X_1 - X_3 + X_5$ have the same ODEs, as well as $X_1 - X_3 - X_5$ and $X_1 + X_3 - X_5$.

Table 4.3: **Reduced PDEs and ODEs for Symmetry Combinations $X_1 \pm X_3 \pm X_5$**

Case(s)	Reduced PDEs	Reduced ODEs
$X_1 - X_3 + X_5$	$(1-r)F + (z_2 + z_1)F_{z_1} + (rz_2 + z_2)F_{z_2} + \frac{1}{2}z_2^2\sigma^2 F_{z_2 z_2} = 0$	$(1-r)g + (1-rw + w\sigma^2)g' + \frac{1}{2}w^2\sigma^2 g'' = 0$
$X_1 - X_3 - X_5$	$(-1-r)F + (z_2 + z_1)F_{z_1} + (rz_2 + z_2)F_{z_2} + \frac{1}{2}z_2^2\sigma^2 F_{z_2 z_2} = 0$	$(-1-r)g + (1-rw + w\sigma^2)g' + \frac{1}{2}w^2\sigma^2 g'' = 0$
$X_1 + X_3 - X_5$	$(-1-r)F + (z_2 - z_1)F_{z_1} + (rz_2 - z_2)F_{z_2} + \frac{1}{2}z_2^2\sigma^2 F_{z_2 z_2} = 0$	

Next we solve eqn (4.30), leading to us getting an invariant solution to our model using Frobenius method [94]. Equation (4.30) can be rewritten in standard form $g''(w) + P(w)g'(w) + Q(w)g(w) = 0$ by dividing through by $\frac{1}{2}w^2\sigma^2$ (assuming $w \neq 0$ and $\sigma \neq 0$)⁴, i.e.,

$$g''(w) + \frac{2(1-rw + w\sigma^2)}{w^2\sigma^2}g'(w) + \frac{2(1-r)}{w^2\sigma^2}g(w) = 0.$$

We have an irregular singular point case. As similarly done in [95], we assume a solution of

⁴ $g''(w) = \frac{d^2g}{dw^2}$ and $g'(w) = \frac{dg}{dw}$

the form

$$g(w) = e^{\frac{a}{w}} Y(w), \quad Y(w) = \sum_{n=0}^{\infty} a_n w^{n+s}, \quad a_0 \neq 0, \quad a_{-1} = a_{-2} = \dots = 0,$$

where s is the indicial exponent to be determined. The derivatives are

$$g'(w) = e^{\frac{a}{w}} \left(-\frac{a}{w^2} Y + Y' \right), \quad (4.33a)$$

$$g''(w) = e^{\frac{a}{w}} \left(\frac{a^2}{w^4} Y + \frac{2a}{w^3} Y - \frac{2a}{w^2} Y' + Y'' \right). \quad (4.33b)$$

Substituting eqns (4.33) into an eqn (4.30) we get

$$\frac{1}{2} w^2 \sigma^2 Y'' + (1 - rw + w\sigma^2 - a\sigma^2) Y' + \left((1 - r) + \frac{a^2 \sigma^2 - 2a}{2w^2} + \frac{ar}{w} \right) Y = 0. \quad (4.34)$$

From the coefficient of w^{s-2} ($n = 0$) we have that

$$\frac{1}{2} \sigma^2 a^2 - a = 0 \implies a = 0 \quad \text{or} \quad a = \frac{2}{\sigma^2}, \quad a_0 \neq 0. \quad (4.35)$$

$a = 0$ is a trivial solution and simplifies the assumed series solution to a regular point Frobenius assumption. We therefore continue with $a = \frac{2}{\sigma^2}$, which when substituted into eqn (4.34) yields

$$\frac{1}{2} w^2 \sigma^2 Y'' + (-1 + w(\sigma^2 - r)) Y' + \left(1 - r + \frac{2r}{w\sigma^2} \right) Y = 0. \quad (4.36)$$

Substituting the derivatives of $Y(w)$ into eqn (4.36) and simplifying we get

$$\begin{aligned} & \sum_{n=1}^{\infty} \left(\frac{1}{2} \sigma^2 (n+s-1)(n+s-2) + (\sigma^2 - r)(n+s-1) + (1-r) \right) a_{n-1} \\ & + \sum_{n=0}^{\infty} \left(\frac{2r}{\sigma^2} - (n+s) \right) a_n = 0. \end{aligned} \quad (4.37)$$

For $n = 0$, w^s gives

$$\left(-s + \frac{2r}{\sigma^2}\right) a_0 = 0 \implies s = \frac{2r}{\sigma^2}, \quad a_0 \neq 0. \quad (4.38)$$

Thus

$$Y(w) = \sum_{n=0}^{\infty} a_n w^{n + \frac{2r}{\sigma^2}}, \quad (4.39)$$

where the recurrence relation for $n \geq 1$ is

$$a_n = -\frac{\frac{1}{2}\sigma^2(n + \frac{2r}{\sigma^2} - 1)(n + \frac{2r}{\sigma^2} - 2) + (\sigma^2 - r)(n + \frac{2r}{\sigma^2} - 1) + (1 - r)}{-n} a_{n-1}. \quad (4.40)$$

Hence the first solution is

$$g_1(w) = C_1 e^{\frac{2}{\sigma^2} w} \sum_{n=0}^{\infty} a_n w^{n + \frac{2r}{\sigma^2}}, \quad (4.41)$$

where C_1 is a constant. The second solution takes the form

$$g_2(w) = g_1(w) \int \frac{e^{-\int P(w) dw}}{g_1(w)^2} dw, \quad (4.42)$$

found using the Wronskian method [96, 97]. Thus we have

$$\begin{aligned} g_2(w) &= g_1(w) \int \frac{e^{-\int P(w) dw}}{g_1^2(w)} dw \\ &= g_1(w) \int \frac{e^{-\int \frac{2}{w^2\sigma^2} - \frac{2r}{w\sigma^2} + \frac{2}{w} dw}}{g_1^2(w)} dw \\ &= g_1(w) \int \frac{1}{g_1^2(w)} \cdot e^{\left(\frac{2}{w\sigma^2} + \frac{2r}{\sigma^2} \ln w - 2 \ln w\right)} dw \\ &= g_1(w) \int \frac{e^{\frac{2}{w\sigma^2} w^{\frac{2r}{\sigma^2} - 2}}}{g_1^2(w)} dw \\ &= C_1 e^{\frac{2}{\sigma^2} w} \sum_{n=0}^{\infty} a_n w^{n + \frac{2r}{\sigma^2}} \int \frac{e^{-\frac{2}{w\sigma^2} w^{\frac{2r}{\sigma^2} - 2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n w^{n + \frac{2r}{\sigma^2}}\right)^2} dw. \end{aligned}$$

The general solution is therefore given by

$$g(w) = C_1 e^{\frac{2}{\sigma^2 w}} \sum_{n=0}^{\infty} a_n w^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2}{w\sigma^2}} w^{\frac{2r}{\sigma^2}-2}}{C_2 \left(\sum_{n=0}^{\infty} a_n w^{n+\frac{2r}{\sigma^2}} \right)^2} dw \right), \quad (4.43)$$

with C_1 and C_2 as constants. Substituting (4.43) into (4.29) we have that

$$F(z_1, z_2) = C_1 e^{\frac{2z_2}{\sigma^2 z_1}} \sum_{n=0}^{\infty} a_n \left(\frac{z_1}{z_2} \right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2z_2}{\sigma^2 z_1}} \left(\frac{z_1}{z_2} \right)^{\frac{2r}{\sigma^2}-2}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{z_1}{z_2} \right)^{n+\frac{2r}{\sigma^2}} \right)^2} d \left(\frac{z_1}{z_2} \right) \right). \quad (4.44)$$

Upon doing another substitution of (4.44) into (4.25) we get the invariant solution

$$v(t, x, y) = C_1 e^{\frac{2x}{\sigma^2 y} + t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x} \right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y}} \left(\frac{y}{x} \right)^{\frac{2r}{\sigma^2}-2}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x} \right)^{n+\frac{2r}{\sigma^2}} \right)^2} d \left(\frac{y}{x} \right) \right). \quad (4.45)$$

This systematic approach, known as the *back substitution method*, generates invariant solutions for all symmetry reductions. Applying this procedure consistently across all symmetry operators produces the complete set of invariant solutions, which we summarize concisely in Table 4.4.

For the cases $X_1 - X_3 - X_5$ and $X_1 + X_3 - X_5$,

$$a_n = -\frac{\frac{1}{2}\sigma^2(n + \frac{2r}{\sigma^2} - 1)(n + \frac{2r}{\sigma^2} - 2) + (\sigma^2 - r)(n + \frac{2r}{\sigma^2} - 1) - (1 + r)}{-n} a_{n-1}, \quad (4.46)$$

while for the other cases its give by eqn (4.40).

Table 4.4: Invariant Solutions for Symmetry Combinations $X_1 \pm X_3 \pm X_5$

Case(s)	Invariant Solutions
$X_1+X_3+X_5$	$Y_1 + Y_2: v(t, x, y) = yC_1 e^{\frac{2x}{\sigma^2 y}} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$
$X_1-X_3+X_5$	$Y_1: v(t, x, y) = C_1 e^{\frac{2x}{\sigma^2 y} + t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$ $Y_1+Y_2: v(t, x, y) = yC_1 e^{\frac{2x}{\sigma^2 y} + 2t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$
$X_1-X_3-X_5$	$Y_1: v(t, x, y) = C_1 e^{\frac{2x}{\sigma^2 y} - t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$ $Y_1 + Y_2: v(t, x, y) = yC_1 e^{\frac{2x}{\sigma^2 y}} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$
$X_1+X_3-X_5$	$Y_1: v(t, x, y) = C_1 e^{\frac{2x}{\sigma^2 y} - t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$ $Y_1+Y_2: v(t, x, y) = yC_1 e^{\frac{2x}{\sigma^2 y} - 2t} \sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}} \left(1 + \int \frac{e^{-\frac{2x}{\sigma^2 y} \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-2}}}{C_2 \left(\sum_{n=0}^{\infty} a_n \left(\frac{y}{x}\right)^{n+\frac{2r}{\sigma^2}}\right)^2} d\left(\frac{y}{x}\right)\right)$

4.2.2 Case 2

Choosing $X_2 + X_3 + X_5$ gives the characteristic equations are:

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dy}{1+y} = \frac{dv}{v}. \quad (4.47)$$

From the first ratio we have that t is a constant. i.e $t = z_1$ for some new variable z_1 .

Taking the second and third ratios we have

$$\begin{aligned} \frac{dx}{x} &= \frac{dy}{1+y}, \\ \Rightarrow z_2 &= \frac{x}{1+y}, \end{aligned} \quad (4.48)$$

where z_2 is some new variable.

The third and fourth ratios give

$$\begin{aligned}\frac{dy}{1+y} &= \frac{dv}{v}, \\ \Rightarrow v &= (1+y)C_2,\end{aligned}\tag{4.49}$$

where C_2 is a constant of integration. The invariants are thus as follows,

$$z_1 = t, \quad z_2 = \frac{x}{1+y}, \quad v(t, x, y) = (1+y)F(z_1, z_2),\tag{4.50}$$

for some function F that depends on the new variables z_1 and z_2 . Substituting the invariants (4.50) into eqn (1.3) gives

$$(z_2 - r)F + (rz_2 - z_2^2)F_{z_2} + \frac{1}{2}\sigma^2 z_2^2 F_{z_2 z_2} + F_{z_1} = 0.\tag{4.51}$$

Equation (4.51) is still a PDE and has to be reduced further into an ODE, and we do so by finding its symmetries which are given as

$$Y_1 = \partial_{z_1}, \quad Y_2 = F\partial_F, \quad Y_b = b(z_1, z_2).\tag{4.52}$$

Invariance Under Y_1

The resulting characteristic equations are

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dF}{0},\tag{4.53}$$

which gives the invariants

$$z_2 = z, \quad F = g(z).\tag{4.54}$$

Substituting the invariants (4.54) into eqn (4.51) we have an ODE

$$(-r + z)g(z) + (rz - z^2)g'(z) + \frac{1}{2}(z)^2\sigma^2g''(z) = 0. \quad (4.55)$$

Invariance Under $Y_1 + Y_2$

The characteristic equations are

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dF}{F}, \quad (4.56)$$

which gives the invariants,

$$z_2 = w, F = e^{z_1}h(w). \quad (4.57)$$

Upon substituting the invariants (4.57) into eqn (4.51) gives an ODE

$$(1 - r + w)h(w) + (rw - w^2)h'(w) + \frac{1}{2}w^2\sigma^2h''(w) = 0. \quad (4.58)$$

Table 4.5 gives a summary of other reductions under both Y_1 and $Y_1 + Y_2$ for the combinations $X_2 \pm X_3 \pm X_5$, with $X_2 + X_3 + X_5$ exclusive.

Table 4.5: Reduced PDEs and ODEs for Symmetry Combinations $\mathbf{X}_2 \pm \mathbf{X}_3 \pm \mathbf{X}_5$

Case(s)	Reduced PDEs	Reduced ODEs
$\mathbf{X}_2 - \mathbf{X}_3 + \mathbf{X}_5$	$(z_2 - r)F + (rz_2 + z_2^2)F_{z_2} + \frac{1}{2}\sigma^2 z_2^2 F_{z_2 z_2} + F_{z_1} = 0$	\mathbf{Y}_1 : $(-r + z)g(z) + (rz + z^2)g'(z) + \frac{1}{2}(z)^2\sigma^2 g''(z) = 0.$ $\mathbf{Y}_1 + \mathbf{Y}_2$: $(1 - r + w)h(w) + (rw + w^2)h'(w) + \frac{1}{2}w^2\sigma^2 h''(w) = 0.$
$\mathbf{X}_2 - \mathbf{X}_3 - \mathbf{X}_5$	$(-z_2 - r)F + (rz_2 + z_2^2)F_{z_2} + \frac{1}{2}\sigma^2 z_2^2 F_{z_2 z_2} + F_{z_1} = 0$	\mathbf{Y}_1 : $(-r - z)g(z) + (rz + z^2)g'(z) + \frac{1}{2}(z)^2\sigma^2 g''(z) = 0.$ $\mathbf{Y}_1 + \mathbf{Y}_2$: $(1 - r - w)h(w) + (rw + w^2)h'(w) + \frac{1}{2}w^2\sigma^2 h''(w) = 0.$
$\mathbf{X}_2 + \mathbf{X}_3 - \mathbf{X}_5$	$(-z_2 - r)F + (rz_2 - z_2^2)F_{z_2} + \frac{1}{2}\sigma^2 z_2^2 F_{z_2 z_2} + F_{z_1} = 0$	\mathbf{Y}_1 : $(-r - z)g(z) + (rz - z^2)g'(z) + \frac{1}{2}(z)^2\sigma^2 g''(z) = 0.$ $\mathbf{Y}_1 + \mathbf{Y}_2$: $(1 - r - w)h(w) + (rw - w^2)h'(w) + \frac{1}{2}w^2\sigma^2 h''(w) = 0.$

Equation (4.55) has a regular singular point. Therefore, an assumed Frobenius series solution is

$$g(z) = \sum_{n=0}^{\infty} a_n z^{n+s}, \quad a_0 \neq 0.$$

Substituting the derivatives and the series $g(z)$ into (4.55) we get

$$\sum_{n=0}^{\infty} \left[\left(\frac{1}{2}\sigma^2(n+s)(n+s-1) + r(n+s) - r \right) a_n + (1 - (n+s))a_{n-1} \right] z^{n+s} = 0. \quad (4.59)$$

Now, to solve for the indicials, we set $n = 0$ and get

$$\left(\frac{1}{2}\sigma^2 s(s-1) + rs - r \right) a_0 = 0.$$

Since $a_0 \neq 0$, the indicial equation is

$$\frac{1}{2}\sigma^2 s(s-1) + rs - r = 0,$$

which when solved yields

$$s_1 = 1, \quad s_2 = -\frac{2r}{\sigma^2}. \quad (4.60)$$

For $s_1 = 1$ so we have the first solution as

$$g_1(z) = z \sum_{n=0}^{\infty} a_n^{(1)} z^n. \quad (4.61)$$

For $s_2 = -\frac{2r}{\sigma^2}$, the second solution is

$$g_2(z) = z^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} z^n. \quad (4.62)$$

The recurrence relation (for $n \geq 1$) is given by

$$a_n = \frac{(n + s - 2)}{r(n + s - 1) + \frac{1}{2}\sigma^2(n + s)(n + s - 1)} a_{n-1}. \quad (4.63)$$

Thus, the general solution is given by

$$g(z) = C_1 z \sum_{n=0}^{\infty} a_n^{(1)} z^n + C_2 z^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} z^n, \quad (4.64)$$

where C_1 and C_2 are constants, and a_n satisfies the recurrence relation (4.63) for $s = s_2$.

Upon performing back substitution, the resulting invariant solution is

$$v(t, x, y) = C_1 x \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1+y} \right)^n + (1+y) C_2 \left(\frac{x}{1+y} \right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1+y} \right)^n. \quad (4.65)$$

We now solve (4.58) by following the same procedure. In standard form we write it as

$$h''(w) + \left(\frac{2r}{\sigma^2 w} - \frac{2}{\sigma^2} \right) h'(w) + \left(\frac{2(1-r)}{\sigma^2 w^2} + \frac{2}{\sigma^2 w} \right) h(w) = 0. \quad (4.66)$$

We have a regular singular point at $w = 0$, thus we continue with the Frobenius method.

Similar to the above procedure, we assume the solution of the form

$$h(w) = \sum_{n=0}^{\infty} a_n w^{n+s}, \quad a_0 \neq 0. \quad (4.67)$$

Finding the derivatives and substituting them along with (4.67) into (4.58) yields

$$\sum_{n=0}^{\infty} \left[\left(\frac{1}{2} \sigma^2 (n+s)(n+s-1) + r(n+s) + (1-r) \right) a_n + (2-n-s) a_{n-1} \right] w^{n+s} = 0. \quad (4.68)$$

For $n = 0$, we have

$$\left(\frac{1}{2} \sigma^2 s(s-1) + rs + (1-r) \right) a_0 = 0 \quad a_0 \neq 0. \quad (4.69)$$

Solving for the indicials we get

$$s_1 = \frac{\sigma^2 - 2r + \sqrt{4r^2 + 4r\sigma^2 + \sigma^4 - 8\sigma^2}}{2\sigma^2}, \quad s_2 = \frac{\sigma^2 - 2r - \sqrt{4r^2 + 4r\sigma^2 + \sigma^4 - 8\sigma^2}}{2\sigma^2}. \quad (4.70)$$

The recurrence relation for $n \geq 1$ is given by

$$a_n = \frac{(n+s-2)}{1 + r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}. \quad (4.71)$$

The general solution is thus as follows

$$h(w) = C_1 w^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} w^n + C_2 w^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} w^n, \quad (4.72)$$

where $a_n^{(1)}$ and $a_n^{(2)}$ satisfy the recurrence relation for $s = s_1$ and $s = s_2$, respectively, and

C_1, C_2 are arbitrary constants. Upon doing back substitution we get an invariant solution

$$v(t, x, y) = (1 + y)e^t C_1 \left(\frac{x}{1 + y} \right)^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1 + y} \right)^n + (1 + y)e^t C_2 \left(\frac{x}{1 + y} \right)^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1 + y} \right)^n. \quad (4.73)$$

Now, for the ODEs in Table 4.5, the indicial equations for invariance under Y_1 are the same as those given in (4.60), and for invariance under $Y_1 + Y_2$, they match those in (4.70). In each case, the denominator of the recurrence relation under invariance by Y_1 is $D_{n_1} = r(n + s - 1) + \frac{1}{2}\sigma^2(n + s)(n + s - 1)$, while under invariance by $Y_1 + Y_2$, the denominator becomes $D_{n_2} = D_{n_1} + 1$. Notably, the recurrence relation for $X_2 - X_3 - X_5$ is the negative of that for $X_2 + X_3 + X_5$, and similarly, the recurrence for $X_2 + X_3 - X_5$ is the negative of that for $X_2 - X_3 + X_5$.

Therefore, the general solutions for invariance under Y_1 have the same structure as in (4.64), while those for invariance under $Y_1 + Y_2$ follow the structure of (4.72). The only distinction lies in the recurrence relation, where s is substituted by the corresponding indicial roots s_1 and s_2 . Table 4.6 provides a detailed summary of these recurrence relations.

Table 4.6: **Recurrence Relations**

Case(s)	Recurrence Relations
$X_2 - X_3 + X_5$	$Y_1: a_n = -\frac{(n+s)}{r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$ $Y_1 + Y_2: a_n = -\frac{(n+s)}{1+r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$
$X_2 - X_3 - X_5$	$Y_1: a_n = -\frac{(n+s-2)}{r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$ $Y_1 + Y_2: a_n = -\frac{(n+s-2)}{1+r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$
$X_2 + X_3 - X_5$	$Y_1: a_n = \frac{(n+s)}{r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$ $Y_1 + Y_2: a_n = \frac{(n+s)}{1+r(n+s-1) + \frac{1}{2}\sigma^2(n+s)(n+s-1)} a_{n-1}.$

Table 4.7 presents the invariant solutions for the remaining combinations of $X_2 \pm X_3 \pm X_5$, excluding the case $X_2 + X_3 + X_5$, which was already addressed in (4.65) and (4.73).

Table 4.7: Invariant Solutions for Symmetry Combinations $X_2 \pm X_3 \pm X_5$

Case(s)	Invariant Solutions
$X_2 - X_3 + X_5$	$Y_1: v(t, x, y) = \frac{C_1 x \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1-y}\right)^n + C_2 \left(\frac{x}{1-y}\right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1-y}\right)^n}{1-y}.$ $Y_1 + Y_2: v(t, x, y) = \frac{e^t \left(C_1 \left(\frac{x}{1-y}\right)^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1-y}\right)^n + C_2 \left(\frac{x}{1-y}\right)^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1-y}\right)^n \right)}{1-y}.$
$X_2 - X_3 - X_5$	$Y_1: v(t, x, y) = C_1 x \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1-y}\right)^n + (1-y) C_2 \left(\frac{x}{1-y}\right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1-y}\right)^n.$ $Y_1 + Y_2: v(t, x, y) = (1-y) e^t C_1 \left(\frac{x}{1-y}\right)^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1-y}\right)^n + (1-y) e^t C_2 \left(\frac{x}{1-y}\right)^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1-y}\right)^n.$
$X_2 + X_3 - X_5$	$Y_1: v(t, x, y) = \frac{C_1 x \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1+y}\right)^n + C_2 \left(\frac{x}{1+y}\right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1+y}\right)^n}{1+y}.$ $Y_1 + Y_2: v(t, x, y) = \frac{e^t \left(C_1 \left(\frac{x}{1+y}\right)^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{1+y}\right)^n + C_2 \left(\frac{x}{1+y}\right)^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{1+y}\right)^n \right)}{1+y}.$

4.2.3 Case 3

Now, if we choose $X_3 + aX_5$, we have the following characteristic equations

$$\frac{dt}{0} = \frac{dx}{x} = \frac{dy}{y} = \frac{dv}{av}, \quad (4.74)$$

which gives the invariants

$$z_1 = t, \quad z_2 = \frac{x}{y}, \quad v(t, x, y) = ayF(z_1, z_2), \quad (4.75)$$

for some function F that depends on the new variables z_1 and z_2 . Substituting the invariants (4.75) into eqn (1.3) gives

$$(z_2 - r)F + (rz_2 - z_2^2)F_{z_2} + \frac{1}{2}\sigma^2 z_2^2 F_{z_2 z_2} + F_{z_1} = 0, \quad (4.76)$$

Equation (4.76), which coincides with equation (4.51), remains a PDE and requires further reduction to an ODE. This is achieved by identifying its symmetries, which are the same as those given in (4.52). Following the same procedure as in Case 4.2.2 we get the following invariant solutions ⁵:

Invariant Solution Under $Y_1 = \partial_{z_1}$

$$v(t, x, y) = aC_1 x \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{y}\right)^n + ayC_2 \left(\frac{x}{y}\right)^{-\frac{2x}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{y}\right)^n. \quad (4.77)$$

Invariant Solution Under $Y_1 + Y_2 = \partial_{z_1} + \partial_{z_2}$

$$v(t, x, y) = aye^t \left(c_1 \left(\frac{x}{y}\right)^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{y}\right)^n + c_2 \left(\frac{x}{y}\right)^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{y}\right)^n \right). \quad (4.78)$$

4.2.4 Case 4

Taking $X_4 + aX_5$, we have the following characteristic equations

$$\frac{dt}{0} = \frac{dx}{2xy} = \frac{dy}{y^2} = \frac{\sigma^2 dv}{v(2x - 2ry + a\sigma^2)}, \quad (4.79)$$

which gives the invariants

$$z_1 = t, \quad z_2 = \frac{x}{y^2}, \quad v(t, x, y) = e^{\left(\frac{2x}{\sigma^2 y} - \frac{2r \ln y}{\sigma^2} - \frac{a}{y} + F(z_1, z_2)\right)}. \quad (4.80)$$

Substituting above invariants (4.80) into eqn (1.3) we have

$$-r + az_2 + rz_2 F_{z_2} + \frac{z_2^2 \sigma^2}{2} F_{z_2}^2 + \frac{z_2^2 \sigma^2}{2} F_{z_2 z_2} + F_{z_1} = 0. \quad (4.81)$$

⁵Where a_n s are the same as (4.63) and (4.71) respectively.

Equation (4.81) must be reduced to an ODE, and it's symmetries are

$$Y_1 = \partial_{z_1}, \quad Y_2 = \partial_F, \quad Y_b = -e^{-F}b(z_1, z_2). \quad (4.82)$$

Invariance Under Y_1

The characteristic equations are

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dF}{0}, \quad (4.83)$$

which when solved we get the invariants

$$z_2 = z \quad \text{and} \quad F(z_1, z_2) = g(z). \quad (4.84)$$

Substituting the invariants (4.84) into (4.81) yields

$$-r + az + rzg'(z) + \frac{z^2\sigma^2}{2}(g'(z))^2 + \frac{z^2\sigma^2}{2}g''(z) = 0. \quad (4.85)$$

The ODE is non-linear, so we can use Riccati method by linearizing the non-linear term through substitutions. First we let $u(z) = g'(z)$. Substituting gives

$$-r + az + rzu + \frac{z^2\sigma^2}{2}u^2 + \frac{z^2\sigma^2}{2}u' = 0.$$

Divide through by $\frac{z^2\sigma^2}{2}$ (assuming $z \neq 0$ and $\sigma \neq 0$)

$$u' + u^2 + \frac{2r}{\sigma^2 z}u = \frac{2(r - az)}{\sigma^2 z^2}.$$

This is a Riccati equation of the form

$$u' + u^2 + P(z)u = Q(z).$$

The standard Riccati transformation is

$$u(z) = -\frac{y'(z)}{y(z)}. \quad (4.86)$$

Substituting (4.86) into (4.2.4) we get

$$-\frac{y''}{y} + \frac{(y')^2}{y^2} + \left(-\frac{y'}{y}\right)^2 + \frac{2r}{\sigma^2 z} \left(-\frac{y'}{y}\right) = \frac{2(r-az)}{\sigma^2 z^2},$$

which simplifies to

$$-\frac{y''}{y} - \frac{2r}{\sigma^2 z} \frac{y'}{y} = \frac{2(r-az)}{\sigma^2 z^2}.$$

Multiply through by $-y$ to eliminate denominators gives

$$y'' + \frac{2r}{\sigma^2 z} y' - \frac{2(r-az)}{\sigma^2 z^2} y = 0. \quad (4.87)$$

Equation (4.87) is now of the Frobenius form. The Frobenius series solution is assumed as

$$y(z) = \sum_{n=0}^{\infty} a_n z^{n+s}, \quad (4.88)$$

and the derivatives are

$$y'(z) = \sum_{n=0}^{\infty} (n+s) a_n z^{n+s-1}, \quad y''(z) = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n z^{n+s-2}. \quad (4.89)$$

Substituting (4.88) and (4.89) into (4.87) gives

$$\sum_{n=0}^{\infty} \left[(n+s)(n+s-1) + \frac{2r}{\sigma^2} (n+s) - \frac{2r}{\sigma^2} \right] a_n z^{n+s-2} + \frac{2a}{\sigma^2} \sum_{n=0}^{\infty} a_n z^{n+s-1} = 0. \quad (4.90)$$

For $n = 0$, from the lowest power terms z^{s-2} for $a_0 \neq 0$ gives

$$s(s-1) + \frac{2r}{\sigma^2}s - \frac{2r}{\sigma^2} = 0. \quad (4.91)$$

From (4.91) we solve for the indicials and get

$$s_1 = 1, \quad s_2 = -\frac{2r}{\sigma^2}. \quad (4.92)$$

For $n \geq 1$, we get the recurrence relation

$$a_n = -\frac{2a}{\sigma^2(n+s)(n+s-1) + 2r(n+s-1)}a_{n-1}. \quad (4.93)$$

For $s_1 = 1$, the first solution is given by

$$y_1(z) = z \sum_{n=0}^{\infty} a_n^{(1)} z^n. \quad (4.94)$$

While for $s_2 = -\frac{2r}{\sigma^2}$ the second solution is given by

$$y_2(z) = z^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} z^n. \quad (4.95)$$

The general solution is

$$y(z) = C_1 y_1(z) + C_2 y_2(z), \quad (4.96)$$

where $y_1(1)$ and $y_2(z)$ are given in (4.94) and (4.95) respectively. Now, $g(z)$ is recovered by integrating the general solution (4.96)

$$u(z) = -\frac{y'(z)}{y(z)} = g'(z), \quad (4.97)$$

which yields

$$g(z) = -\ln y(z) + C, \quad \text{where } C \text{ is a constant of integration.} \quad (4.98)$$

Therefore, the final solution to (4.85) is

$$g(z) = -\ln \left(C_1 z \sum_{n=0}^{\infty} a_n^{(1)} z^n + C_2 z^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} z^n \right) + C. \quad (4.99)$$

The invariant solution is

$$v(t, x, y) = e^{\left(\frac{2x}{\sigma^2 y} - \frac{2r \ln y}{\sigma^2} - \frac{a}{y} + -\ln \left(C_1 \left(\frac{x}{y^2} \right) \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{y^2} \right)^n + C_2 \left(\frac{x}{y^2} \right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{y^2} \right)^n \right) + C \right)}, \quad (4.100)$$

where $a_n^{(1)}$ and $a_n^{(2)}$ are found by substituting the indicials (4.92) into (4.93).

Invariance Under $Y_1 + Y_2$

The resulting characteristic equations are

$$\frac{dz_1}{1} = \frac{dz_2}{0} = \frac{dF}{1}, \quad (4.101)$$

which yield the invariants

$$z_2 = w \quad \text{and} \quad F(z_1, z_2) = z_1 + h(w). \quad (4.102)$$

Substituting the invariants (4.102) into (4.81) yields

$$(1 - r) + aw + rwh'(w) + \frac{w^2 \sigma^2}{2} (h'(w))^2 + \frac{w^2 \sigma^2}{2} h''(w) = 0. \quad (4.103)$$

In order to solve (4.103), the same procedure used to solve for the invariant solution under Y_1 is followed. Transforming an ODE into a Riccati equation gives

$$u' + u^2 + \frac{2r}{\sigma^2 w} u = \frac{2(r-1-aw)}{\sigma^2 w^2}, \quad (4.104)$$

which when linearized leads to

$$y'' + \frac{2r}{\sigma^2 w} y' - \frac{2(r-1-aw)}{\sigma^2 w^2} y = 0. \quad (4.105)$$

Equation (4.105) is of the Frobenius format, so we assume a Frobenius series solution

$$y(w) = \sum_{n=0}^{\infty} a_n w^{n+s}. \quad (4.106)$$

The first and second derivatives are given by

$$y'(w) = \sum_{n=0}^{\infty} (n+s) a_n w^{n+s-1}, \quad (4.107)$$

$$y''(w) = \sum_{n=0}^{\infty} (n+s)(n+s-1) a_n w^{n+s-2}. \quad (4.108)$$

Substituting (4.106), (4.107), and (4.108) into (4.105)

$$\sum_{n=0}^{\infty} \left[(n+s)(n+s-1) + \frac{2r}{\sigma^2} (n+s) - \frac{2(r-1)}{\sigma^2} \right] a_n w^{n+s-2} + \frac{2a}{\sigma^2} \sum_{n=0}^{\infty} a_n w^{n+s-1} = 0. \quad (4.109)$$

The indicial equation for $n=0$ (such that $a_0 \neq 0$) from the lowest power w^{s-2} is

$$s(s-1) + \frac{2r}{\sigma^2} s - \frac{2(r-1)}{\sigma^2} = 0, \quad (4.110)$$

which when solved gives the indicials

$$s_{1,2} = \frac{1 - \frac{2r}{\sigma^2} \pm \sqrt{\frac{4r^2}{\sigma^4} + \frac{4r}{\sigma^2} + 1 - \frac{8}{\sigma^2}}}{2}. \quad (4.111)$$

For $n \geq 1$, the recurrence relation is

$$a_n = -\frac{2a}{\sigma^2 \left((n+s)(n+s-1) + \frac{2r}{\sigma^2}(n+s) - \frac{2(r-1)}{\sigma^2} \right)} a_{n-1}. \quad (4.112)$$

For roots s_1 and s_2 from (4.111):

$$y(w) = C_1 w^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} w^n + C_2 w^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} w^n. \quad (4.113)$$

Recall from (4.97):

$$u(w) = -\frac{y'(w)}{y(w)} = h'(w). \quad (4.114)$$

Integrating we get

$$h(w) = -\ln y(w) + C. \quad (4.115)$$

Thus, the general solution is

$$h(w) = -\ln \left(C_1 w^{s_1} \sum_{n=0}^{\infty} a_n^{(1)} w^n + C_2 w^{s_2} \sum_{n=0}^{\infty} a_n^{(2)} w^n \right) + C. \quad (4.116)$$

The resulting invariant solution is

$$v(t, x, y) = e^{\left(\frac{2x}{\sigma^2 y} - \frac{2r \ln y}{\sigma^2} - \frac{a}{y} + t - \ln \left(C_1 \left(\frac{x}{y^2} \right)^{\sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{y^2} \right)^n} + C_2 \left(\frac{x}{y^2} \right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{y^2} \right)^n + C \right) \right)}. \quad (4.117)$$

CHAPTER 5

RESULTS AND DISCUSSION

In this chapter, we present and analyze the exact invariant solutions derived in chapter 4. These solutions are visualized through a series of plots to examine their behavior under specific parameters. We discuss both the similarities and the distinctions in their graphical profiles, interpret their financial implications, and contextualize the results within the framework of the model PDE.

The following parameter values are used throughout this chapter:

- Constants: $C_1 = 1.0$, $C_2 = 1.0$, $a_0 = 1.0$, number of series terms $n_{\text{terms}} = 10$,
- Running average values: $y \in (90, 120)$.

5.1 Case 4.2.2 & Case 4.2.3 Results

The Y_1 invariant solution

$$v(t, x, y) = y \left[C_1 \frac{x}{y} \sum_{n=0}^{\infty} a_n^{(1)} \left(\frac{x}{y} \right)^n + C_2 \left(\frac{x}{y} \right)^{-\frac{2r}{\sigma^2}} \sum_{n=0}^{\infty} a_n^{(2)} \left(\frac{x}{y} \right)^n \right],$$

offers stability and smooth dependence on $\frac{x}{y}$, ideal for visualizing spatial dynamics across the parameter ranges specified in the chapter introduction. Thus, invariant solutions from case 4.2.3 will be used, specifically with $a = 1$.

To determine appropriate values for the interest rate r and volatility σ , the option value is plotted as a function of these parameters in Figure 5.1.

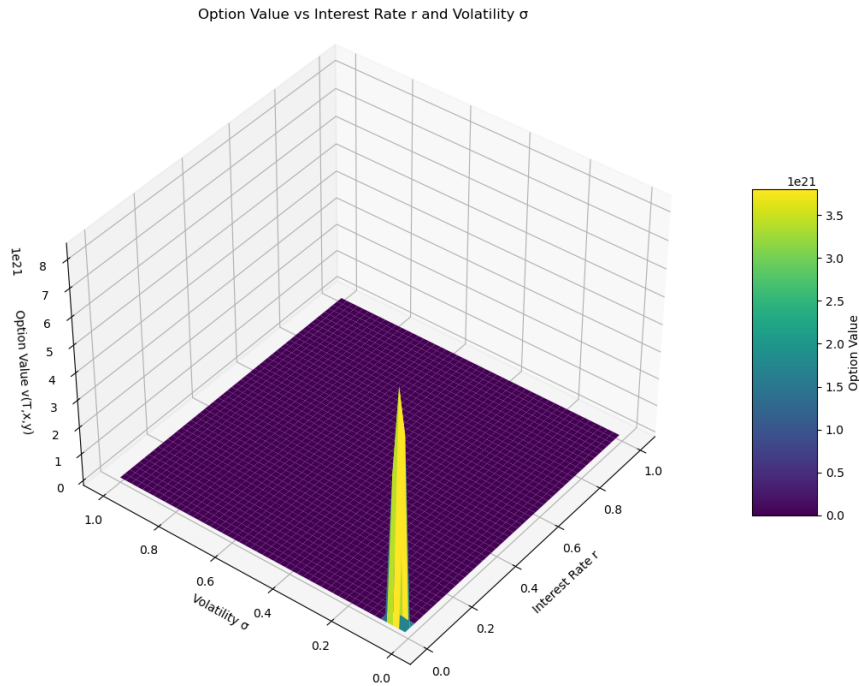


Figure 5.1: Case 4.2.2 & Case 4.2.3 2D and 3D option value (payoff) graphs; for $0 \leq r \leq 1$, $0 < \sigma < 1$, $x = 150$ and $y = 110$.

The figure illustrates that the option value $v(T, x, y)$ remains positive for relatively low values of both the interest rate and volatility, approximately within the ranges $0.01 < r < 0.1$ and

$0.01 < \sigma < 0.15$. Consequently, for the subsequent analysis where either r or σ is held constant, we select $r = \sigma = 0.09$ as representative parameter values.

5.1.1 OTM and ITM Conditions

In this sub-subsections, we present a graphical analysis of the option value at maturity ($t = T$), along with an investigation of the effects of the risk-free interest rate and volatility on the option's valuation.

Option Value vs Strike Price K and Running Average y

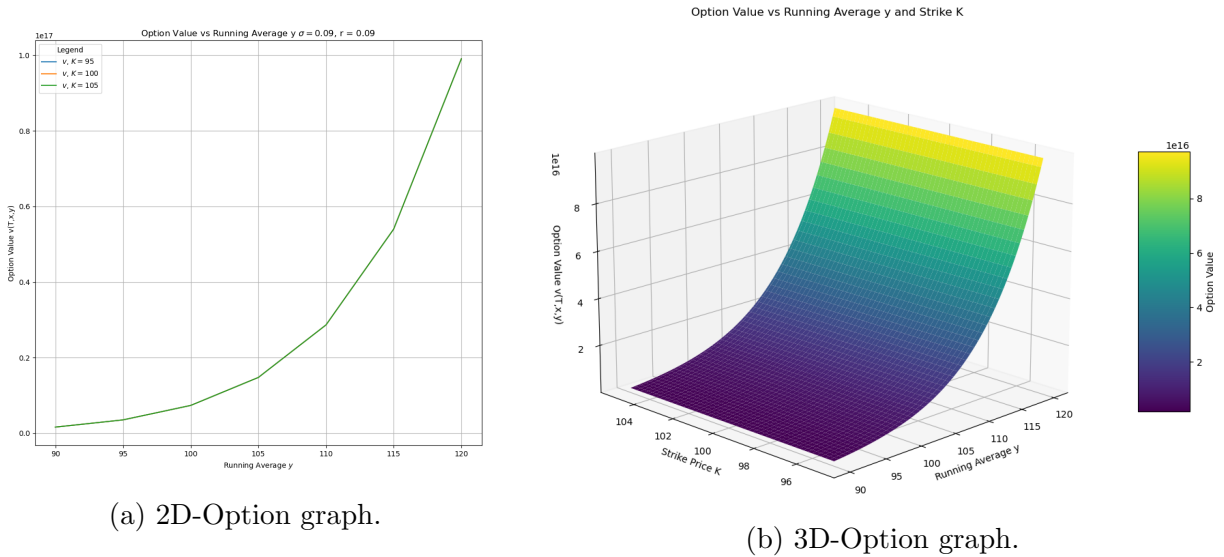


Figure 5.2: Case 4.2.2 & Case 4.2.3 2D and 3D Option value graphs; for $90 \leq y \leq 120$ and $95 \leq K \leq 105$.

Compared to Figure 1.1, Figure 5.2 exhibits a similar overall shape but with a notably smoother profile, reflecting a more stable sensitivity of the option value to changes in the running average price y . The graph demonstrates that the option value $v(T, x, y)$ is zero for low running average values, consistent with the option being out-of-the-money and therefore worthless at maturity. Conversely, the option value increases monotonically as y increases, indicating a positive payoff showing that an option is in-the-money. This behavior aligns

with the fundamental characteristics of an arithmetic Asian call option, where the payoff depends on the average underlying price exceeding the strike. The smoothness of the curve further suggests stability in the pricing model's response to variations in the running average, reinforcing the robustness of the derived invariant solution.

Option Value vs Interest Rates r and Running Average y

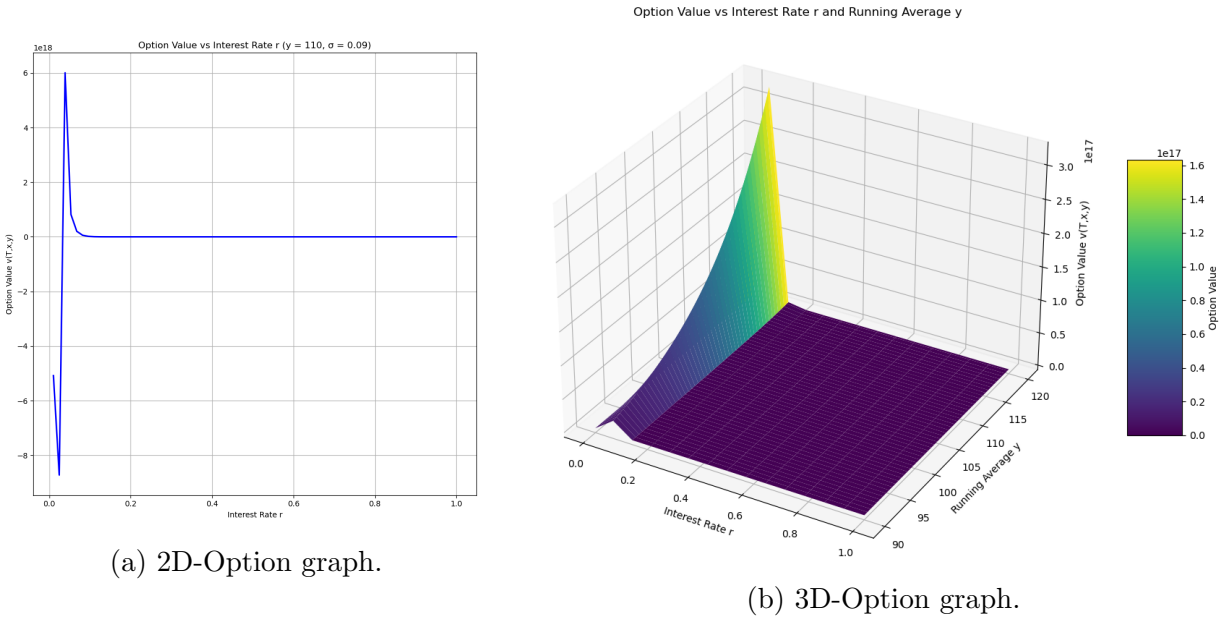


Figure 5.3: Case 4.2.2 & Case 4.2.3 2D and 3D Option value graphs; for $0 < r < 1$ and $90 \leq y \leq 120$.

In Figure 5.3a, we observe that for interest rates $r \approx \in (0, 0.051)$, the option value satisfies $v(T, x, y) < 0$, indicating losses at maturity. For $r \approx \in (0.051, 0.054)$, the option value increases steeply, reaching a maximum near $r \approx 0.054$. Thereafter, the value converges to zero as $r \rightarrow 1$. Figure 5.3b presents a 3D visualization of these dynamics.

Option Value vs Volatility σ and Running Average y

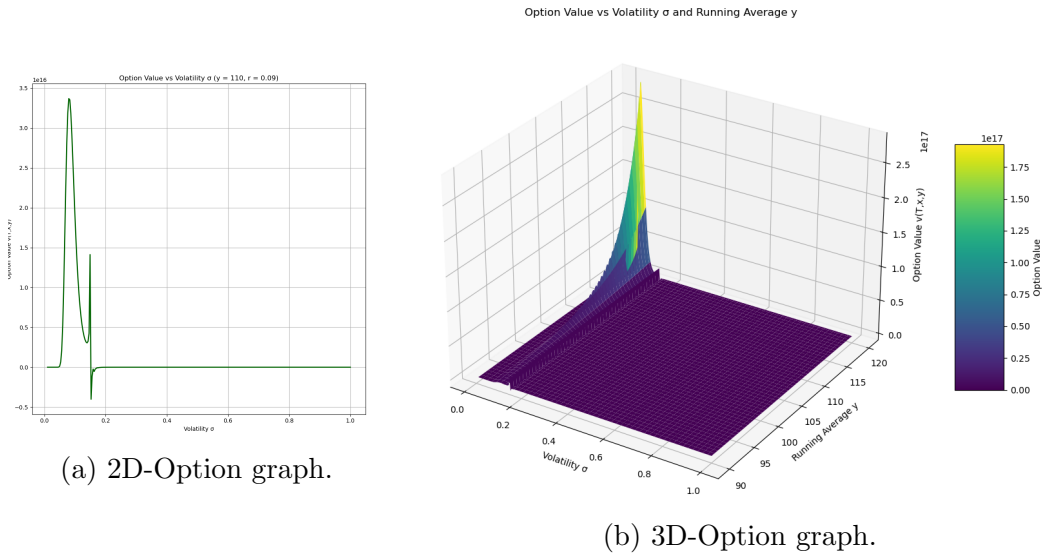
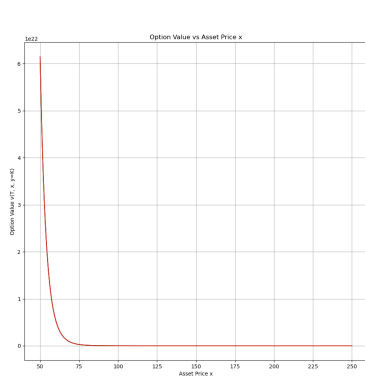


Figure 5.4: Case 4.2.2 & Case 4.2.3 2D and 3D Option value graphs; for $0 < \sigma < 1$ and $90 \leq y \leq 120$.

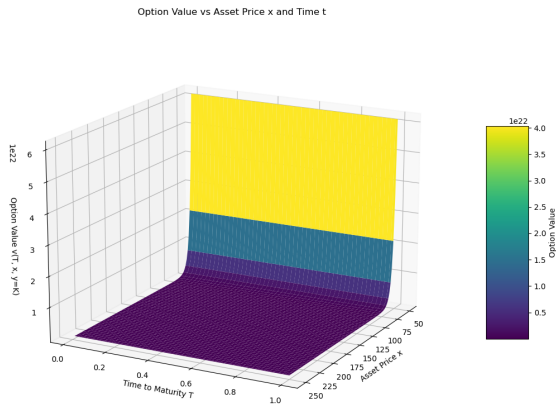
Figure 5.4a shows that the option value fluctuates before eventually declining to zero. Losses occur for volatility levels approximately in the range $\sigma \in (0.151, 0.173)$, while the maximum payoff is observed near $\sigma \approx 0.08$.

5.1.2 ATM Condition

From the plots in Figure 5.5, when the running average y equals the strike price K , the option retains value for underlying asset prices x approximately in the interval $(50, 75)$, before converging toward zero. This behavior is analogous to that observed in Figure 1.2, where the option's payoff approaches zero as time t approaches maturity T . However, in our case, the derived invariant solution is independent of time. Notably, the option maintains positive value even when the current underlying asset price is below the strike price, within the specified range.



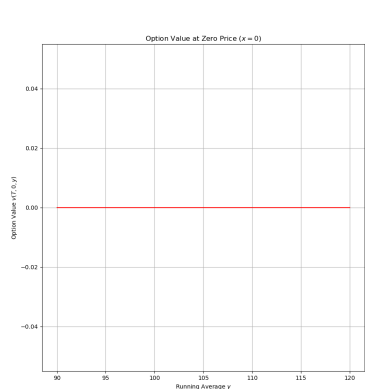
(a) 2D-Option graph.



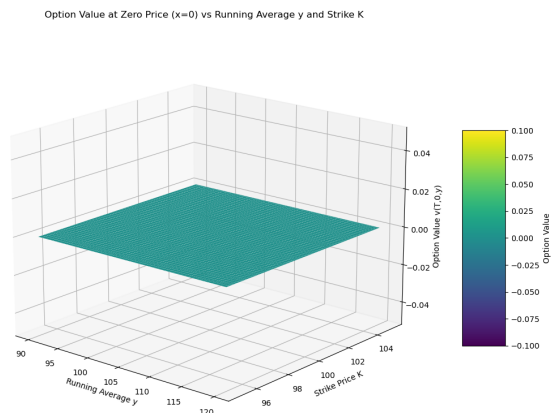
(b) 3D-Option graph.

Figure 5.5: Case 4.2.2 & Case 4.2.3 2D and 3D Option value graphs; for $0 \leq t \leq 1$, $50 \leq x \leq 250$, and $K = 110$.

5.1.3 Z-P Condition



(a) 2D-Option graph.



(b) 3D-Option graph.

Figure 5.6: Case 4.2.2 & Case 4.2.3 2D and 3D Option value graphs; for $0 \leq t \leq 1$ and $90 \leq y \leq 120$.

If the underlying asset price x drops to zero, the option becomes worthless, yielding a payoff of zero. Figure 1.3 illustrates that under such conditions, the option's value may exhibit a residual effect influenced by factors such as the risk-free interest rate, time to maturity, the running average of the asset price, and the strike price. However, in our specific scenario, no such residual value (or discounting effect) is observed.

The preceding graphs correspond to invariant solutions derived under the symmetry generator Y_1 . However, when considering invariant solutions associated with the combined symmetry $Y_1 + Y_2$, and using the same parameter values, the resulting plots are devoid of meaningful financial interpretation. Specifically, the option values exhibit nonphysical behavior, ranging from negative values up to zero. Even under elevated interest rates and volatility, these solutions fail to reproduce any resemblance to the benchmark profiles illustrated in Figure 1.1. Figure 5.7 depicts this behavior in 2D, with varying numbers of series terms n_{terms} in the expansion coefficients a_n .

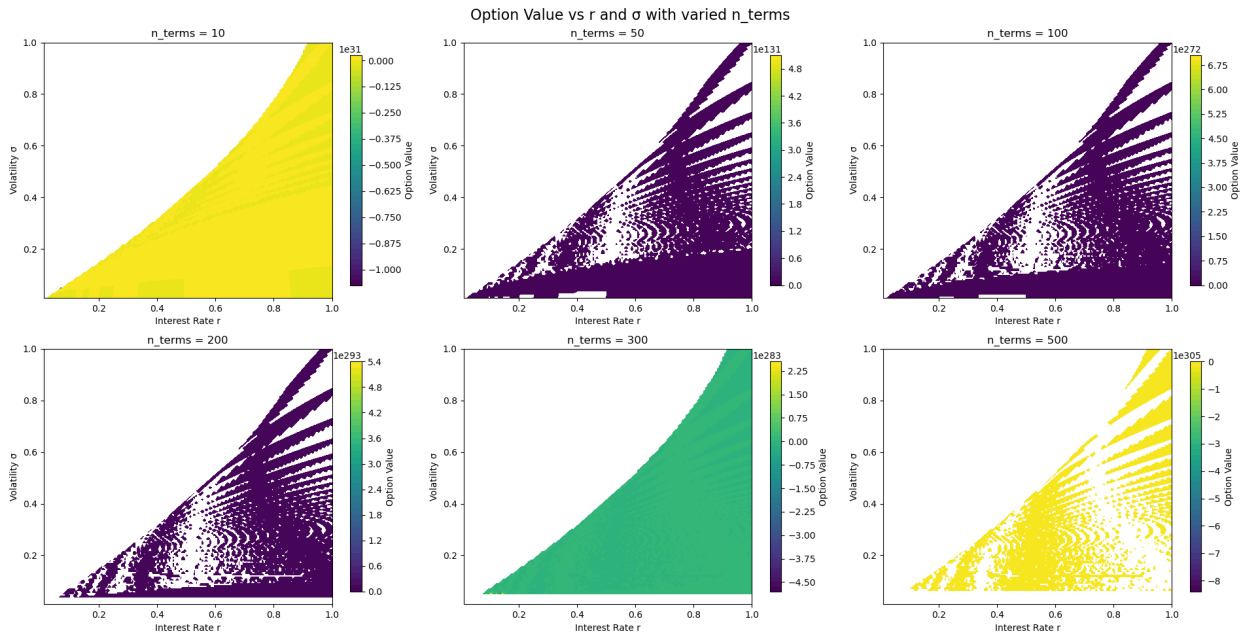


Figure 5.7: Case 4.2.2 & Case 4.2.3 2D Option value graphs for invariant solutions under $Y_1 + Y_2$; for $y = 110$, $x = 150$, and $t = 1$.

5.1.4 Conclusion

The invariant solutions derived for cases 4.2.2 and 4.2.3, as analyzed in section 5.1, establish a rigorous mathematical framework for modeling option value dynamics under a variety of market scenarios. A central feature of these solutions is the shared similarity variable $\frac{x}{f(y)}$, where $f(y) = 1 \pm y$ or y , which induces a power-law scaling and generates consistent graphical behavior across both two-dimensional and three-dimensional visualizations, as seen in Figures 5.2, 5.3, 5.4, 5.5, and 5.6.

However, a critical limitation was observed in case 4.2.2 when adopting the transformation $f(y) = 1 - y$, which resulted in negative option values even under standard market inputs. Such outputs are not financially admissible, as they contradict the fundamental no-arbitrage condition and violate the principle of limited liability inherent in option pricing. As a result, only the transformation $f(y) = 1 + y$ is retained for its economic validity, particularly under the symmetry combinations $X_2 + X_3 \pm X_5$, which yield financially interpretable solutions.

In addition, an attempt to explore invariant solutions under the combined generator $Y_1 + Y_2$ revealed significant analytical challenges. Despite identical parameter settings, the resulting solutions produced degenerate option values, with the graphs consistently indicating a value of zero regardless of variations in macroeconomic parameters such as interest rate or volatility.

The option pricing surfaces consistently exhibit monotonic increases in value for running averages exceeding the strike price (in-the-money conditions), with option values tending to zero when out-of-the-money ($y < K$), which aligns well with classical financial theory. Sensitivity analysis reveals that the option value is particularly responsive to moderate levels of interest rates ($r \approx 0.054$) and volatility ($\sigma \approx 0.08$), beyond which option value rapidly decays as $r, \sigma \rightarrow 1$. This behavior is clearly illustrated in Figures 5.3 and 5.4. These observations

justify the use of $r = \sigma = 0.09$ in all subsequent simulations, ensuring financial interpretability.

From a practical financial perspective, an option holder would prefer market dynamics that steer both interest rates and volatility toward these empirically favorable regions by maturity, thus maximizing the terminal payoff.

5.2 Case 4.2.4 Results

Following the same procedure as in section 5.1, start by finding the interest rates and volatility values using Figure 5.8. Thereafter, we provide the graphs in 2D and 3D with short interpretations and give the general conclusion to the section. The interest rates that give positive

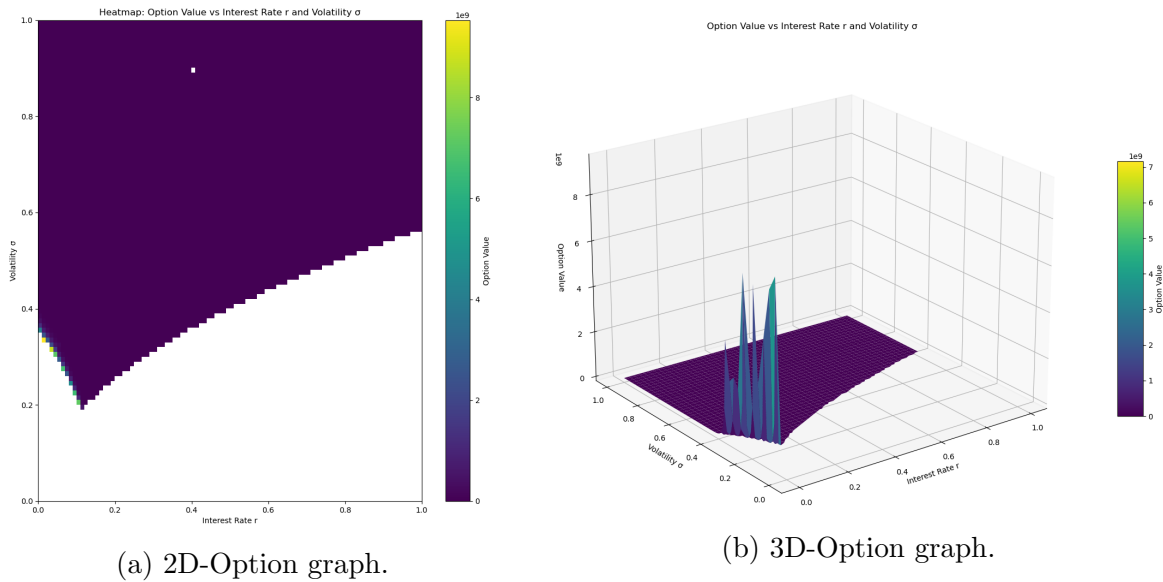
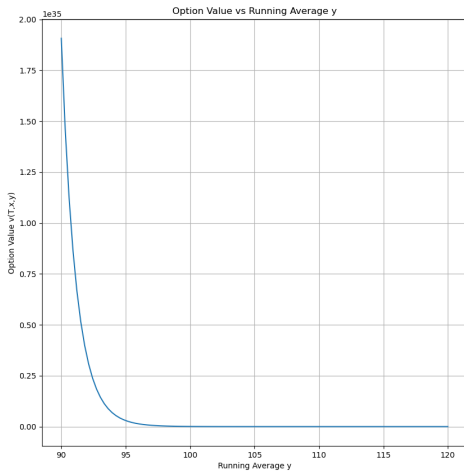


Figure 5.8: Case 4.2.4 2D and 3D Option value (payoff) graphs; for $0 \leq r \leq 1$, $0 < \sigma < 1$, $x = 150$ and $y = 110$.

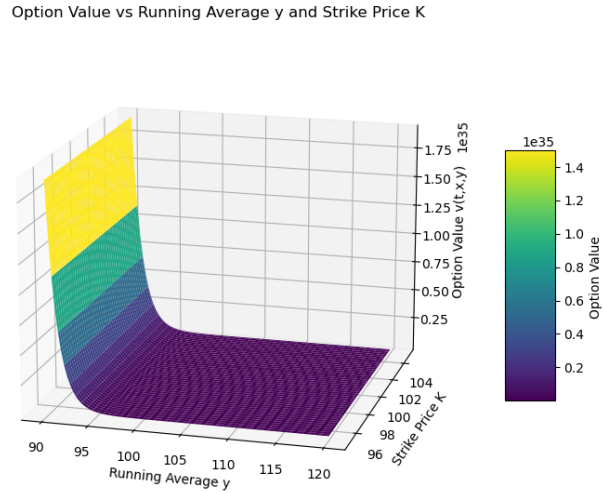
payoff are $r \approx \in (0.004, 0.108)$ and volatility $\sigma \approx \in (0.198, 0.352)$. We will use $r = 0.09$ and $\sigma = 0.2$.

5.2.1 OTM and ITM Conditions

Option Value vs Strike Price K and Running Average y



(a) 2D-Option graph.



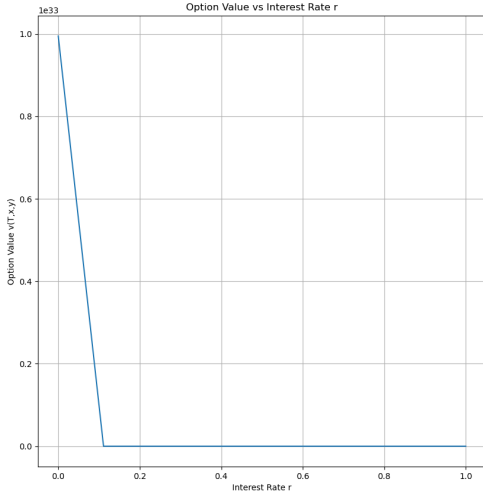
(b) 3D-Option graph.

Figure 5.9: Case 4.2.4 2D and 3D Option value graphs; for $90 \leq y \leq 120$ and $95 \leq K \leq 105$.

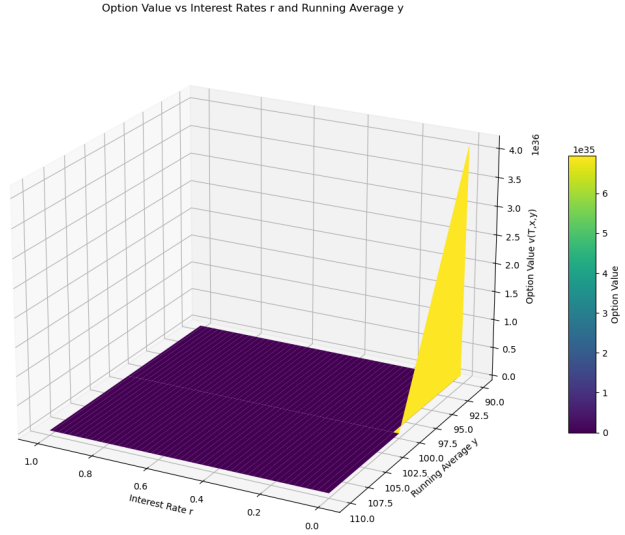
Figure 5.9 demonstrates that the option value declines rapidly to zero as the running average y increases. The contract maintains positive value primarily when $y \in (90, 97)$, indicating the option is in-the-money in this region, which is not what is expected in comparison to Figure 1.1.

Option Value vs Interest Rates r and Running Average y

As seen in Figure 5.10, the option retains value for interest rates $r \in (0, 0.11)$, with peak payoff observed around $r = 0.03$. The sensitivity to the running average remains consistent with the previous findings, reinforcing the ITM threshold.



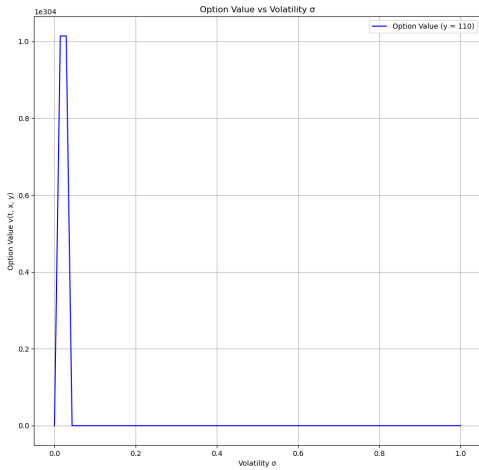
(a) 2D-Option graph.



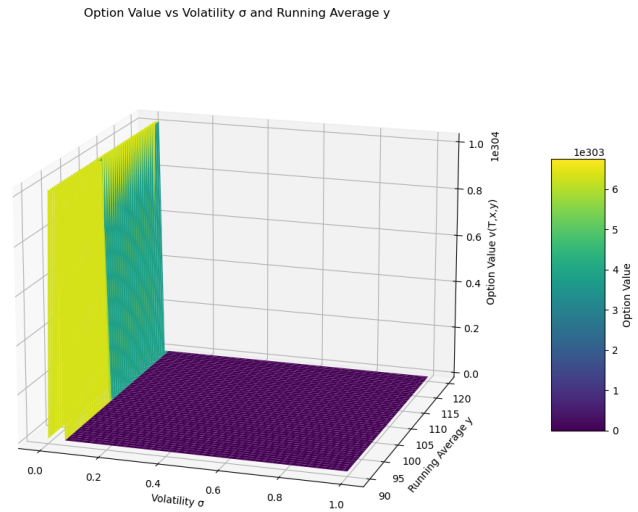
(b) 3D-Option graph.

Figure 5.10: Case 4.2.4 2D and 3D Option value graphs; for $0 < r < 1$ and $90 \leq y \leq 120$.

Option Value vs Volatility σ and Running Average y



(a) 2D-Option graph.

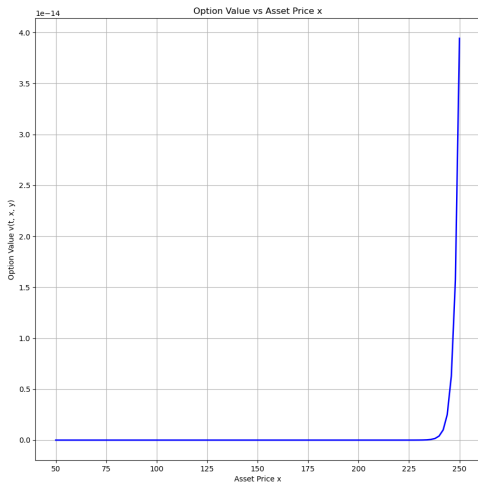


(b) 3D-Option graph.

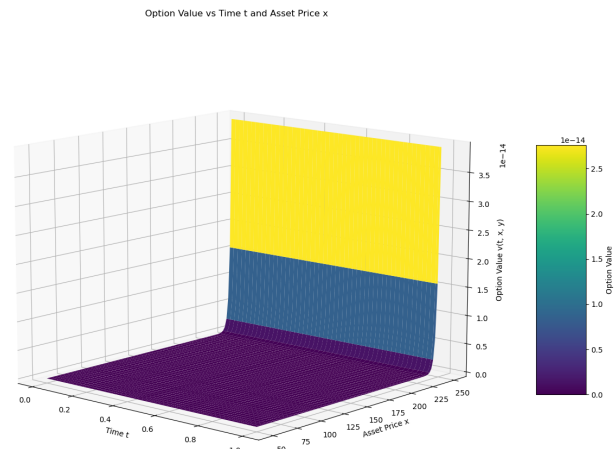
Figure 5.11: Case 4.2.4 2D and 3D Option value graphs; for $0 < \sigma < 1$ and $90 \leq y \leq 120$.

Figure 5.11 reveals that the option exhibits positive value only for low volatility levels, specifically $\sigma \in (0, 0.044)$, with maximum payoff at $\sigma \approx 0.019$. This illustrates the contract's aversion to volatility in this regime.

5.2.2 ATM Condition



(a) 2D-Option graph.

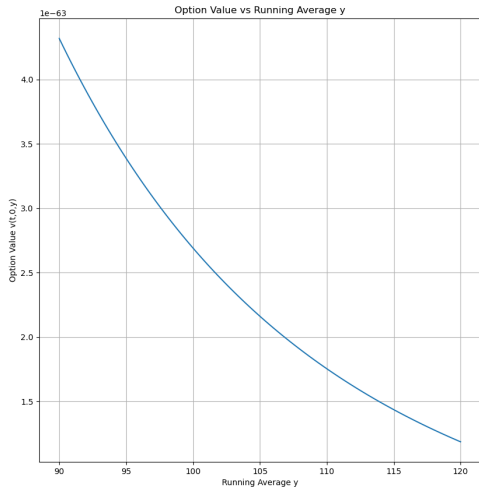


(b) 3D-Option graph.

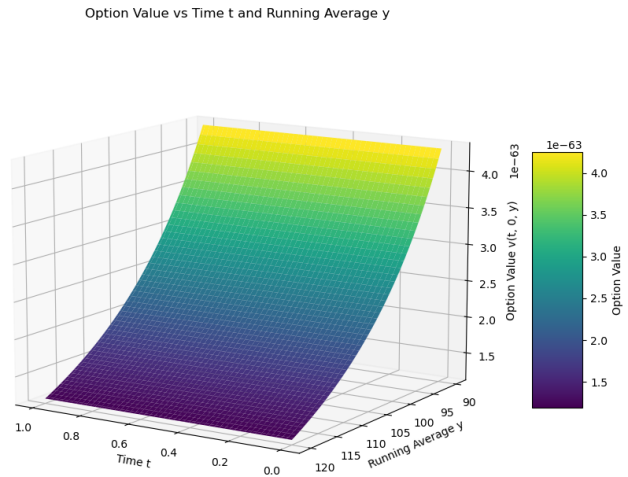
Figure 5.12: Case 4.2.4 2D and 3D Option value graphs; for $0 \leq t \leq 1$, $50 \leq x \leq 250$, and $y = K = 110$.

From Figure 5.12, when the strike price K equals the running average $y = 110$, the option becomes valuable only when the current asset price $x \in (235, 250)$, although very low and becomes negligible (on the order of 10^{-14}).

5.2.3 Z-P Condition



(a) 2D-Option graph.



(b) 3D-Option graph.

Figure 5.13: Case 4.2.4 2D and 3D Option value graphs; for $0 \leq t \leq 1$ and $90 \leq y \leq 120$.

Figure 5.13 shows that when the asset price tends toward zero, the computed payoff becomes negligible (on the order of 10^{-16}), effectively yielding zero value, thus satisfying the zero-price boundary condition.

5.2.4 Conclusions

The analysis in section 5.2 demonstrates a consistent and interpretable behavior of the option value across varying financial parameters. Regions where the option is in-the-money are distinctly characterized by lower running averages, moderate interest rates, and reduced volatility. The results show that the option is highly sensitive to the running average y . Notably, interest rates below approximately 11% and volatilities below 4.4% are conducive to positive payoffs in this case. The at-the-money and zero-price conditions further validate the theoretical boundary behaviors expected from such contracts.

CHAPTER 6

DISCUSSION AND CONCLUSION

This dissertation set out to find exact analytical solutions for arithmetic Asian options using Lie symmetry methods, addressing the lack of full double reductions in existing literature. The aim and objectives were largely achieved. The determining equations were found and solved, infinitesimal generators were derived, an optimal system was constructed, and numerous invariant solutions were obtained through systematic reductions of the governing PDE to ODEs.

Among the cases explored, valid invariant solutions emerged most clearly for scenarios where the reduced ODEs admit Frobenius-type expansions that converge meaningfully for practical parameter ranges. In particular, cases 4.2.2 and 4.2.3 produced solutions that are consistent with established financial principles as discussed in section 5.1.4, resulting in bounded and interpretable option value graphs for realistic volatility and interest rate levels.

However, this research revealed some inherent limitations, especially for case 4.2.1 invariant solutions, which exhibit severe exponential blow-up near the singularity, notably the factor $\exp\left(\frac{2x}{\sigma^2 y} + t\right)$, that becomes unbounded as volatility converges to zero, i.e $\sigma \rightarrow 0$. Additionally, the recurrence relations for coefficients a_n yield unstable terms, hindering meaningful series sums. Integral expressions from the Wronskian method often contain singularities or converge slowly, making numerical evaluation difficult. Together, these issues limit practical computational use and robust plotting across domains. Moreover, the effect of time on option value was not studied, as meaningful invariant solutions showed no time dependence.

Despite these challenges, the findings mark an important step forward, as this work demonstrates that full double reductions and explicit invariant solutions for Asian options are not only possible but can be systematically derived and classified. This represents significant progress in bridging rigorous symmetry analysis with applied financial modeling.

Future work should extend this framework to put options and more complex payoffs like barrier and multi-asset options to increase applicability. Combining analytical Lie symmetry solutions with numerical methods could resolve singularity and stability issues, enabling reliable pricing across diverse market conditions. Applying these methods to stochastic volatility and other path-dependent derivatives may yield valuable new benchmarks.

In conclusion, this dissertation confirms that Lie symmetry methods provide exact solutions for arithmetic Asian call options. While visualization challenges exist at low volatility, key cases offer strong benchmarks. The research connects rigorous symmetry analysis with practical pricing, establishing a foundation for improved exotic derivative models and risk strategies.

BIBLIOGRAPHY

- [1] Jonathan E Ingersoll. *Theory of financial decision making*, volume 3. Rowman & Littlefield, 1987.
- [2] Jin E Zhang. Pricing continuously sampled Asian options with perturbation method. *Journal of Futures Markets: Futures, Options, and Other Derivative Products*, 23(6):535–560, 2003.
- [3] Éric Briys. *Options, futures and exotic derivatives : theory, application and practice*. Wiley frontiers in finance. Wiley, 1998.
- [4] Peter Carr, Christian-Oliver Ewald, and Yajun Xiao. On the qualitative effect of volatility and duration on prices of Asian options. *Finance Research Letters*, 5(3):162–171, 2008.
- [5] John C. Hull and Sankarshan Basu. *Options, futures, and other derivatives*. Pearson Education India, 2016.
- [6] Paul Wilmott. *Paul Wilmott on quantitative finance*. John Wiley & Sons, 2013.

- [7] James Chen. What is an Asian option? how they work vs. standard options, 2022. Updated April 21, 2022. Reviewed by Somer Anderson. Fact checked by Ariel Courage.
- [8] Jan Vecer. Black–Scholes representation for Asian options. *Mathematical Finance*, 24(3):598–626, 2014.
- [9] Jiaying Han and Yicheng Hong. Review of Asian options. *Open Access Library Journal*, 9(2):1–8.
- [10] Maryam Baghestani, Esmaeil Pishbahar, and Ghader Dahsti. The pricing of Asian options using Monte Carlo simulation (case study: Soybean meal). *Agricultural Economics*, 12(3):1–26, 2018.
- [11] Thomas E. Copeland, John Fred Weston, Kuldeep Shastri, et al. *Financial theory and corporate policy*, volume 4. Pearson Addison Wesley Boston, 2005.
- [12] Michael J. Brennan and Eduardo S. Schwartz. Evaluating natural resource investments. *The Journal of Business*, 58(2):135–157, 1985.
- [13] Vadim Linetsky. Spectral expansions for Asian (average price) options. *Operations Research*, 52(6):856–867, 2004.
- [14] Xiao Jing Cai, Shuairu Tian, Nannan Yuan, and Shigeyuki Hamori. Interdependence between oil and East Asian stock markets: Evidence from wavelet coherence analysis. *Journal of International Financial Markets, Institutions and Money*, 48:206–223, 2017.
- [15] Juraj Hruška et al. Delta-gamma-theta hedging of crude oil Asian options. *Acta Universitatis Agriculturae et Silviculturae Mendelianae Brunensis*, 63(6):1897–1903, 2015.
- [16] Reinval Jaakko, Kilpinen Samu, Kärki Markus, and Rintamäki Tuomas. The pricing of Asian commodity options. In *Aalto University School of Science, Seminar on Case Studies in Operations Research, Mat-2.4177*, 2014.

- [17] Sigurdur Helgason. Sophus Lie, the mathematician. *ResearchGate Article*, 1994. Available at <https://www.researchgate.net/publication/251269525>.
- [18] Sophus Lie. Zur theorie des integrabilitätsfaktors. *Christiania Forh.*, pages 242–254, 1874. Reprinted in *Gesammelte Abhandlungen*, Vol. III, no. XIII, pp. 176–187, Teubner, Leipzig, 1922.
- [19] Peter J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, New York, 1986.
- [20] Élie Cartan. A centenary: Sophus Lie. In F. Le Lionnais, editor, *Great Currents of Mathematical Thought*, pages 262–267. Dover, New York, 1971.
- [21] Élie Cartan. Les systèmes de pfaff à cinq variables et les équations aux dérivées partielles du second ordre. *Annales de l'École Normale Supérieure*, 27:109–192, 1910.
- [22] Élie Cartan. Sur l'équivalence absolue de certains systèmes d'équations différentielles et sur certaines familles de courbes. *Bulletin de la Société Mathématique de France*, 42:12–48, 1914.
- [23] Chen Ning Yang. Symmetry and physics. *Proceedings of the American Philosophical Society*, 140(3):267–288, 1996.
- [24] Jürgen Fuchs and Christoph Schweigert. *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*. Cambridge University Press, 2003.
- [25] Tukur A. Sulaiman, Abdullahi Yusuf, Fairouz Tchier, Mustafa Inc, F.M.O. Tawfiq, and F. Bousbahi. Lie-bäcklund symmetries, analytical solutions and conservation laws to the more general $(2 + 1)$ -dimensional boussinesq equation. *Results in Physics*, 22:103850, 2021.

- [26] Mohamed R. Ali, Wen-Xiu Ma, and R. Sadat. Lie symmetry analysis and invariant solutions for (2+1) dimensional Bogoyavlensky-Konopelchenko equation with variable-coefficient in wave propagation. *Journal of Ocean Engineering and Science*, 7(3):248–254, 2022.
- [27] Jean-François Ganghoffer, Vincent Magnenet, and Rachid Rahouadj. Relevance of symmetry methods in mechanics of materials. *Journal of Engineering Mathematics*, 66:103–119, 2010.
- [28] O’Hara John G. Caister, Nicolette C. and Keshlan S. Govinder. Solving the Asian option pde using Lie symmetry methods. *International Journal of Theoretical and Applied Finance*, 13(08):1265–1277, 2010.
- [29] Shipeng Zhou and Liuqing Xiao. An application of symmetry approach to finance: Gauge symmetry in finance. *Symmetry*, 2(4):1763–1775, 2010.
- [30] A. Paliathanasis. Lie symmetries of differential equations in financial mathematics. *Journal of Nonlinear Mathematical Physics*, 23(4):529–538, 2016.
- [31] Clarinda Nhangumbe, E. Fredericks, and Betuel Canhanga. *Lie Symmetry Analysis on Pricing Weather Derivatives by Partial Differential Equations*, pages 875–901. 06 2020.
- [32] Rafail K. Gazizov and Nail H. Ibragimov. Lie symmetry analysis of differential equations in finance. *Nonlinear Dynamics*, 17(4):387–407, 1998.
- [33] Nail H. Ibragimov. *Introduction to Modern Group Analysis*. Kreativ, Tay, Scotland, 2000.
- [34] Chi-Fai Lo and Cho-Hoi Hui. Valuation of financial derivatives with time-dependent parameters: Lie-algebraic approach. 2001.

- [35] Chi-Fai Lo and Cho-Hoi Hui. Pricing multi-asset financial derivatives with time-dependent parameters—Lie algebraic approach. *International Journal of Mathematics and Mathematical Sciences*, 32(7):401–410, 2002.
- [36] Tomas Björk and Camilla Landén. On the term structure of futures and forward prices. In *Mathematical Finance—Bachelier Congress 2000: Selected Papers from the First World Congress of the Bachelier Finance Society, Paris, June 29–July 1, 2000*, pages 111–149. 2002.
- [37] Chao Yue and Chuanhe Shen. Lie symmetry analysis for the fractal bond-pricing model of mathematical finance. *Journal of Mathematics*, 2024(1):9926131, 2024.
- [38] Realeboha Ramoeletsi. Application of Lie symmetries to solving fractional Black-Scholes option pricing model in financial mathematics. Masters thesis, National University of Lesotho, 2022.
- [39] Bosiu C. Kaibe and John G. O’Hara. Symmetry analysis of an interest rate derivatives pde model in financial mathematics. *Symmetry*, 11(8), 2019.
- [40] Bosiu C Kaibe. *Application of Lie Symmetries to Solving Partial Differential Equations associated with the Mathematics of Finance*. PhD thesis, University of Essex, 2021.
- [41] Robert C Merton. Theory of rational option pricing. *The Bell Journal of economics and management science*, pages 141–183, 1973.
- [42] Andrei Shleifer and Robert W Vishny. The limits of arbitrage. *The Journal of finance*, 52(1):35–55, 1997.
- [43] William F Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of finance*, 19(3):425–442, 1964.

- [44] John Lintner. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets: A reply. *The review of economics and statistics*, pages 222–224, 1969.
- [45] Jack L. Treynor. Market value, time, and risk (August 8, 1961). Available at SSRN: <https://ssrn.com/abstract=2600356> or <http://dx.doi.org/10.2139/ssrn.2600356>.
- [46] Eugene F Fama and Kenneth R French. Common risk factors in the returns on stocks and bonds. *Journal of financial economics*, 33(1):3–56, 1993.
- [47] Stephen A Ross. The arbitrage theory of capital asset pricing. *Journal of Economic Theory*, 13(3):341–360, 1976.
- [48] Richard Roll and Stephen A Ross. An empirical investigation of the arbitrage pricing theory. *The journal of finance*, 35(5):1073–1103, 1980.
- [49] Muhammad U. Faruque. An empirical investigation of the arbitrage pricing theory in a frontier stock market: evidence from bangladesh. *Indian Journal of Economics and Business*, 10(4):443–465, December 2011.
- [50] Eugene F Fama and Kenneth R French. A five-factor asset pricing model. *Journal of financial economics*, 116(1):1–22, 2015.
- [51] Mark M Carhart. On persistence in mutual fund performance. *Journal of Finance*, 52(1):57–82, 1997.
- [52] Tatang Ary Gumanti, Bambang Sutrisno, Andreas Andreas, and Denny Bernardus. Empirical study of Fama-French three-factor model and Carhart Four-factor Model in Indonesia.(July 15, 2017). Available at SSRN: <https://ssrn.com/abstract=3314684> or <http://dx.doi.org/10.2139/ssrn.3314684>.

- [53] Darrell Duffie. *Dynamic asset pricing theory*. Princeton University Press, 2010.
- [54] Stanley R Pliska. A history of options. 2010.
- [55] Gorica Malešević. Options: Historical overview, fundamental terminology and valuation techniques. *SVAROG*, (17):149–159, 2018.
- [56] Jun Chen and Ying Wang. Option pricing under stochastic volatility. *Journal of Derivatives*, 27(1):32–47, 2019.
- [57] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.
- [58] A. G. Z. Kemna and A. C. F. Vorst. A pricing method for options based on average asset values. *Journal of Banking & Finance*, 14(1):113–129, 1990.
- [59] Walter Mudzimbabwe. *Pricing methods for Asian options*. PhD thesis, PhD thesis, PHD thesis, University of the Western Cape, 2009.
- [60] Phelim P Boyle. Options: A Monte Carlo approach. *Journal of Financial Economics*, 4(3):323–338, 1977.
- [61] R. Zvan, Peter Forsyth, and K.R. Vetzal. Robust numerical methods for pde models of asian options. *Journal of Computational Finance*, 1, 10 1996.
- [62] Jeffrey Dewynne and William Shaw. Differential equations and asymptotic solutions for arithmetic asian options: 'black-scholes formulae' for asian rate calls. *European Journal of Applied Mathematics*, 19(4):353–391, 2008.
- [63] Zieneb Ali Elshegmani, Rokiah Rozita Ahmad, Saiful Hafiza Jaaman, and Roza Hazli Zakaria. Transforming arithmetic asian option pde to the parabolic equation with constant coefficients. *International Journal of Mathematics and Mathematical Sciences*, 2011:1–6, 2011. Open access article.

- [64] NC Caister, KS Govinder, and JG O'Hara. Optimal system of lie group invariant solutions for the asian option pde. *Mathematical methods in the applied sciences*, 34(11):1353–1365, 2011.
- [65] W. Mudzimbabwe, K.C. Patidar, and P.J. Witbooi. A reliable numerical method to price arithmetic Asian options. *Applied Mathematics and Computation*, 218(22):10934–10942, 2012.
- [66] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.
- [67] George W Bluman and Sukeyuki Kumei. *Symmetries and differential equations*, volume 81. Springer Science & Business Media, 2013.
- [68] Peter J Olver. Symmetry groups and group invariant solutions of partial differential equations. *Journal of Differential Geometry*, 14(4):497–542, 1979.
- [69] Peter J Olver. Evolution equations possessing infinitely many symmetries. *Journal of Mathematical Physics*, 18(6):1212–1215, 1977.
- [70] Peter J Olver. How to find the symmetry group of a differential equation. *Group Theoretic Methods in Bifurcation Theory*, pages 200–239, 2006.
- [71] Nail H. Ibragimov. *CRC handbook of Lie group analysis of differential equations*, volume 3. CRC press, 1995.
- [72] Peter J Olver. *Applications of Lie groups to differential equations*, volume 107. Springer Science & Business Media, 1993.
- [73] Lawrence Dresner. *Applications of Lie's theory of ordinary and partial differential equations*. CRC Press, 1998.

- [74] Maile Khati and Motlatsi Molati. Lie symmetry analysis of a pseudoparabolic pde: Power law in diffusion coefficient with constant viscosity. *Partial Differential Equations in Applied Mathematics*, 4:100060, 2021.
- [75] Motlatsi Molati and Chaudry Masood Khalique. Lie group analysis of a forced KdV equation. *Mathematical Problems in Engineering*, 2013(1):845843, 2013.
- [76] Eman Ali Hussain and Zainab Mohammed Alwan. Solution second-order of partial differential equations by Lie group. *Sci. Int. (Lahore)*, 30(6):919–924, November-December 2018.
- [77] M.E. Oduor Okoya, N. Omolo Ongati, T. J. O. Aminer, and J. Nyakinda Otula. Symmetry group approach to the solution of generalized Burgers equation: $u_t + uu_x = \lambda u_{xx}$. *Applied Mathematical Sciences*, 7(15):717–725, 2013.
- [78] Motlatsi Molati and Chaudry Masood Khalique. Lie symmetry analysis of the time-variable coefficient B-BBM equation. *Advances in Difference Equations*, 2012:1–8, 2012.
- [79] D Levi, L Martina, and P Winternitz. Lie-point symmetries of the discrete liouville equation. *Journal of Physics A: Mathematical and Theoretical*, 48(2), dec 2014.
- [80] Cesar A Gómez Sierra, Motlatsi Molati, and Motlatsi P Ramollo. Exact solutions of a generalized KdV–mKdV equation. *Int. J. Nonlinear Sci*, 13(1):94–98, 2012.
- [81] Mosito Lekhooana and Motlatsi Molati. Nonlinear long waves in shallow water for normalized boussinesq equations. *Results in Physics*, 59:1–2, 2024.
- [82] Mosito Lekhooana, Motlatsi Molati, and Celestin Wafo Soh. Jordan canonical forms for systems of elliptic equations. *Journal of Computational Mathematics and Data Science*, 1:3–4, 2021.

- [83] Motlatsi Molati and Hideki Murakawa. Exact solutions of nonlinear diffusion-convection-reaction equation: A Lie symmetry analysis approach. *Communications in Nonlinear Science and Numerical Simulation*, 67:253–263, 2019.
- [84] Motlatsi Molati. *Group Classification of Coupled Partial Differential Equations with Applications to Flow in a Collapsible Channel and Diffusion Processes*. PhD thesis, University of the Witwatersrand, 2010.
- [85] Nail Khairullovich Ibragimov. *Elementary Lie Group Analysis and Ordinary Differential Equations*, volume 197. Wiley, New York, 1999.
- [86] Chaudry Masood Khalique and Tanki Motsepa. Lie symmetries, group-invariant solutions and conservation laws of the Vasicek pricing equation of mathematical finance. *Physica A: Statistical Mechanics and its Applications*, 505:871–879, 2018.
- [87] Kai-Seng Chou, Guan-Xin Li, and Changzheng Qu. A note on optimal systems for the heat equation. *Journal of Mathematical Analysis and Applications*, 261(2):741–751, 2001.
- [88] W. Sinkala, P.G.L. Leach, and J.G. O’Hara. An optimal system and group-invariant solutions of the Cox-Ingersoll-Ross pricing equation. *Applied Mathematics and Computation*, 201(1):95–107, 2008.
- [89] Amjad Hussain, Shahida Bano, Ilyas Khan, Dumitru Baleanu, and Kottakkaran Sooppy Nisar. Lie symmetry analysis, explicit solutions and conservation laws of a spatially two-dimensional Burgers–Huxley equation. *Symmetry*, 12(1), 2020.
- [90] Mosito Lekhooana. Lie symmetry analysis and conservation laws of systems of elliptic equations. Master’s thesis, National University of Lesotho, 2021.
- [91] H. S. Tang, K. Y. Chong, and B. H. Kee. Lie symmetries, optimal system, and invariant solutions of the generalized Cox-Ingersoll-Ross equation. In *Proceedings of the Interna-*

- tional Conference on Mathematical Sciences and Statistics 2022 (ICMSS 2022)*, pages 103–113. Atlantis Press, 2022.
- [92] Wolfram Research, Inc. *Mathematica, Version 7.0*. Champaign, IL, 2008.
- [93] S. Dimas and D. Tsoubelis. SYM: A new symmetry-finding package for Mathematica. In N. H. Ibragimov, C. Sophocleous, and P. A. Damianou, editors, *Proceedings of the 10th International Conference in Modern Group Analysis*, pages 64–70, Nicosia, 2005. University of Cyprus.
- [94] Asadullah Torabi and Mohammad Alim Rohani. Frobenius method for solving second-order ordinary differential equations. *Journal of Applied Mathematics and Physics*, 8(7):1269–1277, 2020.
- [95] Hung Cheng. Singular points of ordinary differential equations: Lecture eight – irregular singular points of ordinary differential equations. https://math.mit.edu/classes/18.305/WWW_2004_HungCheng/eight1.pdf, 2004. Last accessed: July 2, 2025.
- [96] David B. Slavsky. Class notes: Frobenius method. <http://dslavsk.sites.luc.edu/courses/other/classnotes/frobenius.pdf>. last accessed: July 2, 2025.
- [97] Tsogtgerel Gantumur. Frobenius method. <https://www.math.mcgill.ca/gantumur/math315w14/downloads/frobenius.pdf>, 2014. Last accessed: July 2, 2025.