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Exact solution of a mathematical model for human muscular motion

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ABSTRACT

An ordinary differential equation (ODE) which models human muscular movement is considered. A functional form of the model parameter is specified through the Lie symmetry approach, yielding a different expression from the one derived in the previous study (Kosugi et al., 2019). The Lie point symmetries corresponding to the model parameter are employed for derivation of exact solution.

1. Introduction

In most applications nonlinear differential equations (DEs) are widely used to model various phenomena in the sciences and engineering disciplines. The analytical solutions of DEs provide a means to validate the numerical simulations and experimental observations. The mathematical model under consideration is used to describe the movement of a human finger, θ, namely (see [1])

$$I\ddot{\theta}(t) + \mu\dot{\theta}(t) - \tau(\theta(t)) = 0, \quad t > 0 \tag{1}$$

where $I, \mu \in \mathbb{R}^+$ denote the moment of inertia and coefficient of viscosity respectively. The model parameter $\tau(\theta)$ represents the torque.

In Ref. [1], numerical and experimental techniques were used to solve Eq. (1) with a mathematical expression for $\tau(\theta)$ derived using the principle of virtual work (the resulting equation is (1.2) investigated in [1]). The sufficient condition is determined for Lyapunov stability of an equilibrium of Eq. (1.2) leading to numerical simulation and experimental results. This work employs an analytical technique, Lie symmetry method [2–5], to specify the form of the model parameter $\tau(\theta)$ in (1) and obtain the corresponding solution thereof. Many DEs models for real life applications contain parameters or arbitrary functions which may only be determined through experiments or known physical laws, in most cases the forms of these parameters or functions are assumed. The Lie symmetry approach has proven to be effective in specifying the forms of model parameters.

2. Lie symmetry analysis

We seek the generator of Lie point symmetries of (1) having the form

$$X = \xi(t,\theta)\frac{\partial}{\partial t} + \eta(t,\theta)\frac{\partial}{\partial \theta}. \tag{2}$$

The operator (2) is the generator of symmetries provided

$$X^{[2]}\left(I\ddot{\theta} + \mu\dot{\theta} - \tau(\theta)\right)\Big|_{(1)} = 0, \tag{3}$$

where $X^{[2]}$ is the second prolongation of X given by

$$X^{[2]} = \xi\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial \theta} + \zeta_1\frac{\partial}{\partial \dot{\theta}} + \zeta_2\frac{\partial}{\partial \ddot{\theta}}\,. \tag{4}$$

The coefficients ζ's are defined by

$$\zeta_1 = \eta_t + \dot{\theta}(\eta_\theta - \xi_t) - \dot{\theta}^2\xi_\theta,$$
$$\zeta_2 = \eta_{tt} + \dot{\theta}(2\eta_{t\theta} - \xi_{tt}) + \dot{\theta}^2(\eta_{\theta\theta} - 2\xi_{t\theta}) - \dot{\theta}^3\xi_{\theta\theta} + \ddot{\theta}(\eta_\theta - 2\xi_t - 3\dot{\theta}\xi_\theta).$$

2.1. Lie symmetry classification

Thus, we obtain from (3) the determining equations

$$\xi_{\theta\theta} = 0, \tag{5}$$
$$2\mu\xi_\theta - 2I\xi_{t\theta} + I\eta_{\theta\theta} = 0, \tag{6}$$
$$\tau\eta_\theta - \tau_\theta\eta + I\eta_{tt} + \mu\eta_t - 2\tau\xi_t = 0, \tag{7}$$
$$2I\eta_{t\theta} - I\xi_{tt} + \mu\xi_t - 3\tau\xi = 0. \tag{8}$$

From Eqs. (5) and (6) we have

$$\xi = \theta\, a(t) + b(t), \quad \eta = \theta^2\left(\dot{a}(t) - \frac{\mu a(t)}{I}\right) + \theta c(t) + d(t), \tag{9}$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are integration constant functions. Hence, Eqs. (7) and (8) become

$$\left[\theta^2\left(\frac{\mu a}{I} - \dot{a}\right) - \theta c - d\right]\frac{d\tau}{d\theta} + \left(c - 2\dot{b} - \frac{2\theta\mu a}{I}\right)\tau$$
$$+ \theta^2\left(I\ddot{a} - \frac{\mu^2\dot{a}}{I}\right) + \theta(\mu\dot{c} + I\ddot{c}) + \mu\dot{d} + I\ddot{d} = 0, \tag{10}$$

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$$3a\tau + 3\theta\left(\mu\dot{a} - I\ddot{a}\right) + I\left(\ddot{b} - 2\dot{c}\right) - \mu\dot{b} = 0. \tag{11}$$

Eqs. (10) and (11) are the classifying relations. If $\tau(\theta)$ is arbitrary, then the operator $X = \partial/\partial t$ constitutes the principal Lie algebra and the additional operators are determined by solving Eqs. (10) and (11). In analysing the classifying equations taking into account (9), we note from (11) that τ is linear in θ. This implies that Eq. (1) reduces to a linear second-order ODE with constant coefficients which can be easily solved using some elementary techniques. Therefore, the analysis is restricted to the nonlinear dependence of τ on θ. That is, we consider the case

$$\frac{d^2\tau}{d\theta^2} \neq 0.$$

Upon analysing the classifying relations we obtain

$$\tau(\theta) = k\theta^2 - \frac{6\mu^2\theta}{25I} \tag{12}$$

for an arbitrary nonzero constant k. Therefore, Eq. (1) takes the form

$$I\ddot{\theta}(t) + \mu\dot{\theta}(t) - k\theta^2 + \frac{6\mu^2\theta}{25I} = 0. \tag{13}$$

The corresponding Lie point symmetries

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = 5I\exp\left(\frac{\mu t}{5I}\right)\frac{\partial}{\partial t} - 2\mu\exp\left(\frac{\mu t}{5I}\right)\theta\frac{\partial}{\partial\theta} \tag{14}$$

constitute a two-dimensional symmetry Lie algebra of Eq. (13).

Remark. It is noted that the expression for $\tau(\theta)$ given by (12) is not the same as the one obtained in [1] (see Eq. (1.2) therein).

2.2. Exact solution

The characteristic system for the linear combination $X_1 + X_2$ of (14) is

$$\frac{dt}{1 + 5I\exp\left(\frac{\mu t}{5I}\right)} = \frac{d\theta}{-2\mu\exp\left(\frac{\mu t}{5I}\right)\theta}. \tag{15}$$

Solving (15) we have

$$\theta(t) = \frac{K_1}{\left[1 + 5I\exp\left(\frac{\mu t}{5I}\right)\right]^2} \tag{16}$$

where an integration constant K_1 satisfies the equation

$$25IkK_1 - 6\mu^2 = 0. \tag{17}$$

Therefore, the exact solution of Eq. (13) is given by

$$\theta(t) = \frac{6\mu^2}{25kI\left[1 + 5I\exp\left(\frac{\mu t}{5I}\right)\right]^2}. \tag{18}$$

For graphical representation of solution (18), we use the data $I = 4.2 \times 10^{-3}$kg m^2, $\mu = 0.1$ from [1]. The constant k is determined by imposing the initial conditions (see Figs. 1 and 2).

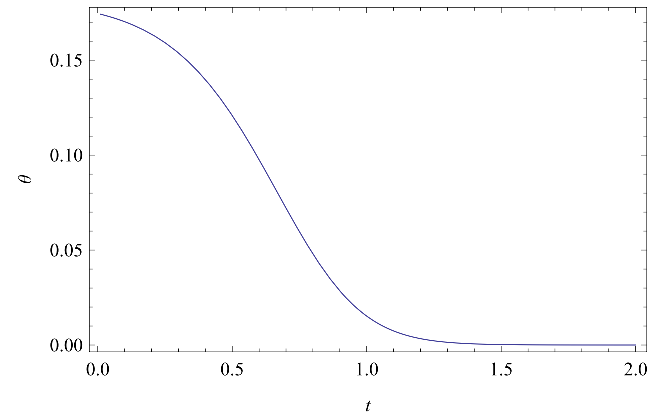

Fig. 1. Solution (18) with initial condition $\theta(0) = \pi/18$ rad.

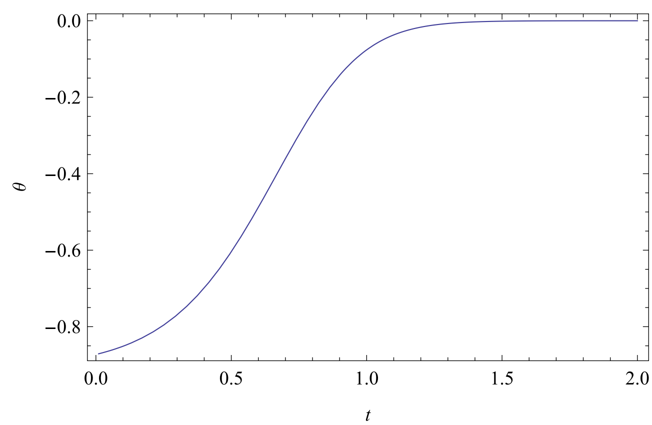

Fig. 2. Solution (18) with initial condition $\theta(0) = -5\pi/18$ rad.

3. Conclusion

The quadratic form of the model parameter, $\tau(\theta)$, was obtained through Lie symmetry analysis and the corresponding symmetry Lie algebra was utilized for construction of exact solution. The exact solution was represented graphically for different initial conditions. It is worth mentioning that the different expressions for $\tau(\theta)$ in the current study and the previous one by Kosugi et al. [1] implies that the submodel (13) is not the same as Eq. (1.2) investigated in [1] and therefore a comparative analysis may not be appropriate. This becomes the subject of future work, to employ Lie symmetry method on Eq. (1.2). The current work is purely out of mathematical interest but the obtained results could be of interest in other research fields such as robotics and physiology because analytical solutions are a benchmark for numerical and experimental results.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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