LIE GROUP ANALYSIS AND INVARIANT SOLUTIONS OF A PSEUDO-PARABOLIC PARTIAL DIFFERENTIAL EQUATION MODELLING SOLVENT UPTAKE IN POLYMERIC SOLIDS

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DECLARATION

I, Maile Khati, student number 200400599, declare that this dissertation submitted for the degree of Master of Science in Applied Mathematics at National University of Lesotho has not previously been submitted by me for a degree at this or any other University. Further, I declare that this is my original work and any work done by others has been acknowledged and referenced accordingly.

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M. Khati

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ABSTRACT

The main purpose of this work is to perform Lie group analysis of a pseudo-parabolic partial differential equation modelling solvent uptake in polymeric solids. Three different sub-models for constitutive laws for the diffusion coefficient and the viscosity will be considered. Once the symmetries have been determined, they will be used to find group-invariant solutions for the optimal systems of one-dimensional subalgebras obtained and where possible, exact solutions will be obtained.

DEDICATION

To my wife, son and the late Professor M. P. Ramollo

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Contents

List of Tables

List of Figures

INTRODUCTION

Products that are commonly referred to as plastics, nylon and deoxyribonucleic acid (DNA) to the general public are known as polymers to chemists. A polymer is a macromolecule which consists of small molecular units that are linked together to form a long chain. The small molecular unit is called a monomer.

If one immerses a polymer sample, for example, polymethylmetacrylate, in an organic solvent such as methanol, solvent particles diffuse into the polymer [8]. In this process an anomalous diffusion behaviour known as Case II which was reported in [1] by Alfrey, Gurnee and Lloyd occurs. Case II diffusion (which occurs in polymer-penetrant systems) explains the process in terms of two basic parameters, the diffusion coefficient and the viscosity of the polymer. It is characterized by linear sorption kinetics and a sharp diffusion front in which the penetrant subsatacially swells the polymer [24]. Alfrey et al documented that as the solvent penetrates into the polymer, swelling takes place and a sharp front is generated between the swollen rubbery shell and inner glassy core at a constant velocity, which result in the growth of the volume of the absorbed liquid to be linear in time. After a certain time the normal (Fickian) diffusion, characterized by the penetrant's volume being proportional to the square root of time, occurs. The Fickian diffusion follows the Fick's laws introduced by Adolf Fick in 1855, which states that diffusion is the movement of components from a high concentration to a low concentration across a concentration gradient, and the concentration changes as a function of time to the change in flux with respect to position.

The most widely known Mathematical model of the uptake of liquids by polymers was proposed by Thomas and Windle [24] in 1982. Case II diffusion and related diffusion process have also been studied by different authors [9, 10, 12, 13]. But recently, research of the model was carried out using pseudo-parabolic differential equation [8].

Problem to be investigated

The pseudoparabolic differential equation modelling solvent uptake in polymeric solids is given by [8]

$$u_t = (D(u)(u + \nu(u)u_t)_x)_x, \tag{1}$$

where $u = u(t, x)$ is the concentration of the solvent, t represents time, x stands for the material coordinates, D is the diffusion coefficient, and ν is the viscosity of the polymer. The solvent has the property that mass is conserved upon penetrating into the polymer, hence $u(t, x)$ satisfies the continuity equation

$$u_t + \text{div } J = 0, \tag{2}$$

where $J = J(t, x)$ describes the flux of the solvent. The flux obeys a generalized Darcy's law of the form

$$J = -D(u)(\nabla u + E(u)\nabla\sigma), \tag{3}$$

where σ is the stress put on the polymer by the solvent particles and E is the stress coefficient.

Equation (1) will be considered for different constitutive laws for the diffusion coefficient $D(u)$ and the viscosity $\nu(u)$. In [8] a proof of (short and long) existence of solutions to the Dirichelt problem for constitutive law $D(u) = u^m$ with $m > 0$ and $\nu(u) = 1$ was given.

Model I:

In this case, it is assumed that there is no phase transition of the polymer during the uptake of the solvent and at least one of the quantities, diffusion coefficient or inverse viscosity, degenerates. For example, take $D(u) = u^m$ with $m > 0$ and $\nu(u) = 1$, then the special case of equation (1) having the form

$$u_t = (u^p u_x + u^q u_{tx})_x, \tag{4}$$

where p and q are real numbers is considered. In the investigation by Hulshof and King [13] using phase plane analysis, it was determined that there exist positive travelling wave solutions with fronts if and only if $0 < m < 2$. It is worth noting that the case $p = q = m$ corresponds to power law in diffusion coefficient with constant viscosity.

Model II:

In this case, the assumption is that there is no phase transition of the polymer during the uptake of the solvent and neither the diffusion coefficient nor the inverse viscosity degenerates in u. We will consider $D(u) = e^{mu}$ and $\nu(u) = e^{-nu}$ where $m, n > 0$ as proposed by Thomas and Windle [24]. Thus the model is given by

$$u_t = (e^{mu}(u + e^{-nu}u_t)_x)_x. \tag{5}$$

Asymptotic analysis on the dependence of the front velocity on m and n was discussed by During and Cohen [9]. Equation (5) was solved using numerical simulations in [24], the move from case II to Fickian diffusion was noted.

Model III:

The asymptotic analysis by Hulshof and King [13] suggested the diffusion coefficient of the form $D(u) = u^\alpha + \epsilon u^\beta$ and the constant viscosity $\nu = 1$ where $\alpha \geq 2$, $0 < \beta < 2$ and $\epsilon << 1$. The term u^α provides for creation of an approximately sharp front and the perturbed term ϵu^β generates a similar front of finite length to cause the sharp front move like a travelling wave. Thus the model is given by

$$u_t = ((u^\alpha + \epsilon u^\beta)(u + u_t)_x)_x. \tag{6}$$

For large time (i.e. $t \to \infty$) equation (6) is approximated to

$$u_t = ((u^\alpha + \epsilon u^\beta)u_x)_x. \tag{7}$$

The classical Lie symmetry analysis approach will be used to solve models (4) and (5) while approximate symmetry analysis of models (6) and (7) will be performed.

Method to be used

In the nineteenth century, Norwegian mathematician Sophus Lie introduced a method of solving differential equations (DEs) known as Lie symmetry (group) analysis which its core idea is based on the invariance of a differential equation under a continuous group of symmetries. A symmetry group of a differential equation is a group of transformation which maps any solution to another solution of a differential equation [20]. In other words, a symmetry of a differential equation is an invertible transformation whose action leaves the DE unchanged. Finding all symmetries is a difficult task as it involves a lot of cumbersome and tedious calculations. But if we consider symmetries that depend continuously on one-parameter and that form a group, we can determine them by using a powerful and versatile systematic procedure known as Lie's algorithm. The Lie's algorithm has been implemented using Computer Algebra Systems(CAS) like Mathematica, Maple, Maxima and Reduce to mention a few [19]. In this work we will use a software package *YaLie* [7] written in Mathematica together with needed algebraic manipulations to determine the Lie point symmetries.

The investigation of the exact solutions plays an important role in the study of PDEs. Many different methods have been developed to find these exact solutions such as the inverse scattering method, Darboux and Bäcklund transformation, Hirota bilinear method, Lie symmetry analysis, e.t.c. However, the latter has proven to be a powerful and systematic approach to construct exact solutions of PDEs by using symmetries to find the invariant solutions. Invariant solutions are then used to reduce the order of PDE and/or reduce PDE to an ordinary differential equation (ODE) which in general is easier to solve. Futhermore, based on the Lie symmetry analysis, many other types of exact solutions can be obtained, such as a travelling wave solutions, soliton solutions, power series solutions, fundamental solutions [21], and so on. Another application of Lie symmetry analysis is the classification of group invariant solutions of DEs by means of optimal system, which provides the minimal list from which all other invariant solutions can be obtained. Thus, invariant solutions can be derived from an optimal system. Hence the optimal system can be used to construct a more abundant group of invariant solutions. In the past six decades there have been considerable developments in the complilation of literature on symmetry analysis of differential equations as evidenced by research papers and books [4, 5, 6, 14, 20, 21, 23] including the references there in.

An outline of the study is as follows. In Chapter 1 a brief theoretical background of Lie symmetry analysis is given. In the remaining chapters, different models of the pseudo-parabolic equation are studied. In Chapters 2 and 3, the Lie point symmetries are obtained for each model and the corresponding optimal system is derived. Thereafter symmetry reductions and group invariant solutions are obtained. In Chapter 4, the approximate symmetries are obtained and the approximate symmetry Lie algebra is used to construct approximate invariant solutions.

Chapter 1

Lie Point Symmetries of Differential Equations

In this chapter we introduce some basic methods of Lie symmetry analysis of differential equations (DEs) that are used throughout this work including the algorithm to determine the Lie point symmetries of PDEs based upon the references [4, 5, 6, 14, 15, 17, 18, 20, 21, 23]. The definitions, theorems, proofs and notations in this chapter are based upon the aforementioned references.

In order for one to clearly get the concepts about Lie symmetry analysis, we will consider a general second-order PDE of the form

$$E(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0 \tag{1.1}$$

in one dependent variable u and two independent variables t and x.

1.1 One parameter group of transformations

Definition 1.1.1. Invertible transformations of the variables t, x, u

$$\bar{t} = f(t, x, u), \quad \bar{x} = g(t, x, u), \quad \bar{u} = h(t, x, u)$$

are called *symmetry transformations* of (1.1) if they leave (1.1) form-invariant (has the same form) in the new variables $\bar{t}, \bar{x}, \bar{u}$, i.e.,

$$E(\bar{t}, \bar{x}, \bar{u}, \bar{u}_t, \bar{u}_x, \bar{u}_{tt}, \bar{u}_{tx}, \bar{u}_{xx}) = 0, \tag{1.2}$$

whenever (1.1) is satisfied.

Definition 1.1.2. A set G of transformations

$$T_a : \bar{t} = f(t, x, u, a), \quad \bar{x} = g(t, x, u, a), \quad \bar{u} = h(t, x, u, a) \tag{1.3}$$

is called a *continuous one-parameter (local) Lie group* of transformations (1.3) in $\mathbb{R}^3$ where f, g, h are differentiable functions and a is a real parameter which continuously takes values in a neighbourhood $\mathcal{D} \subset \mathbb{R}$ of $a = 0$ provided the group properties of closure, identity and inverse are satisfied, i.e.,

(i) **Closure:** If $T_a, T_b \in G$, where $a, b \in \mathcal{D}' \subset \mathcal{D}$ then

$$T_a T_b = T_c \in G, \quad c = \phi(a, b) \in \mathcal{D}.$$

(ii) **Identity:** There exists $T_0 \in G$ such that

$$T_0 T_a = T_a T_0 = T_a,$$

for any $a \in \mathcal{D}' \subset \mathcal{D}$. T_0 is known as the identity of the group.

(iii) **Inverse:** There exists $T_a^{-1} = T_{a^{-1}} \in G$, $a^{-1} \in \mathcal{D}$ such that

$$T_a^{-1} = T_{a^{-1}} = T_0,$$

for any $T_0 \in G$, $a \in \mathcal{D}' \subset \mathcal{D}$. T_a^{-1} is known as the inverse of the group.

The group property (i) can be written as

$$\begin{aligned} \bar{\bar{t}} &= f(\bar{t}, \bar{x}, \bar{u}, b) = f(f(t, x, u, a), b) = f(t, x, u, \phi(a, b)), \\ \bar{\bar{x}} &= g(\bar{t}, \bar{x}, \bar{u}, b) = g(g(t, x, u, a), b) = g(t, x, u, \phi(a, b)), \\ \bar{\bar{u}} &= h(\bar{t}, \bar{x}, \bar{u}, b) = h(h(t, x, u, a), b) = h(t, x, u, \phi(a, b)). \end{aligned} \tag{1.4}$$

The analytic function ϕ is called the group composition law.

For the rest of the study, we will use the term "a one-parameter group" to mean a continuous one-parameter (local) group.

Corrolary 1.1.1. Any one-parameter group of transformation is abelian, i.e.,

$$T_a T_b = T_b T_a.$$

Definition 1.1.3. A group parameter a is canonical if the group composition law is additive, i.e., $\phi(a,b) = a + b$.

If a is canonical, then

$$T_b T_a = T_{a+b} = T_{b+a} = T_a T_b.$$

Theorem 1.1.1. Given an arbitrary composition law $\phi(a,b)$ the canonical parameter $\tilde{a}$ is defined by

$$\tilde{a} = \int_0^a \frac{da}{A(a)},$$

where

$$A(a) = \left.\frac{\partial \phi(a,b)}{\partial b}\right|_{b=0}.$$

Definition 1.1.4. The transformations (1.3) form a *symmetry group* G of E if its invariant form (1.2) is satisfied whenever equation (1.1) holds.

1.2 Infinitisemal generator and Lie's equation

According to the Lie theory the construction of the symmetry group G is equivalent to the determination of its infinitesimal transformations

$$\bar{t} \approx t + a\xi^1(t,x,u), \quad \overline{x} \approx x + a\xi^2(t,x,u), \quad \overline{u} \approx u + \eta(t,x,u) \tag{1.5}$$

obtained by expanding (1.3) in powers of a around $a = 0$ (in a neighborhood $a = 0$ identity) and setting

$$\left.\frac{\partial f(t,x,u,a)}{\partial a}\right|_{a=0} = \xi^1(t,x,u), \quad \left.\frac{\partial g(t,x,u,a)}{\partial a}\right|_{a=0} = \xi^2(t,x,u), \quad \left.\frac{\partial h(t,x,u,a)}{\partial a}\right|_{a=0} = \eta(t,x,u).$$

The components ξ^1, ξ^2 and η are called the infinitesimals of (1.3). The vector (ξ^1, ξ^2, η) is the tangent vector at the point (t,x,u) to the surface curve described by transformed points $(\bar{t}, \overline{x}, \overline{u})$, and it is therefore called the tangent vector field of the group G.

One can now introduce the symbol X (after Lie) of the infinitesimal transformations (1.5) by writing

$$\bar{t} \approx (1+X)t, \quad \bar{x} \approx (1+X)x, \quad \bar{u} \approx (1+X)u, \tag{1.6}$$

where

$$\begin{aligned} X &= (\xi^1(t,x,u), \xi^2(t,x,u), \eta(t,x,u)).\nabla \\ &= \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}, \end{aligned} \tag{1.7}$$

and the operator ∇ is the gradient vector operator $\nabla = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial u}\right)$.

The operator (1.7) is known as the infinitesimal operator or generator of the group G.

Remark 1.2.1. $X((t,x,u)) = (\xi^1(t,x,u), \xi^2(t,x,u), \eta(t,x,u))$.

Theorem 1.2.1. Given infinitesimal transformations (1.5), or an infinitesimal operator (1.7) the transformations (1.3) of the corresponding group G are determined by solving the Lie equations

$$\frac{d\bar{t}}{da} = \xi^1(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{x}}{da} = \xi^2(\bar{t},\bar{x},\bar{u}), \quad \frac{d\bar{u}}{da} = \eta(\bar{t},\bar{x},\bar{u}), \tag{1.8}$$

subject to the initial conditions

$$f(t,x,u,a)|_{a=0} = t, \quad g(t,x,u,a)|_{a=0} = x, \quad h(t,x,u,a)|_{a=0} = u.$$

A one-parameter Lie group of transformation is equivalent to its infinitesimal operator, this allows to represent the solution of Lie equations in terms of Taylor series (exponential map)

$$\bar{t} = \exp(aX)t, \quad \bar{x} = \exp(aX)x, \quad \bar{u} = \exp(aX)u, \tag{1.9}$$

where

$$\exp(aX) = 1 + aX + \frac{a^2}{2!} + \cdots = \sum_{j=0}^{\infty} \frac{a^j}{j!}X^j \tag{1.10}$$

is known as the *Lie series operator*.

1.3 Invariants

Definition 1.3.1. A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point if it remains unchanged by every transformation of a group G, i.e.,

$$(\bar{t}, \overline{x}, \overline{u}) = (f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = (t, x, u), \quad \forall a \in \mathcal{D}' \subset \mathcal{D}. \tag{1.11}$$

Theorem 1.3.1. A point $(t, x, u) \in \mathbb{R}^3$ is an invariant point of a group G with operator

$$X = \xi^1(t, x, u)\frac{\partial}{\partial t} + \xi^2(t, x, u)\frac{\partial}{\partial x} + \eta(t, x, u)\frac{\partial}{\partial u} \tag{1.12}$$

if and only if

$$\xi^1(t, x, u) = \xi^2(t, x, u) = \eta(t, x, u). \tag{1.13}$$

Definition 1.3.2. A function $F(t, x, u)$ is called an invariant of a group G if and only if

$$F(\bar{t}, \overline{x}, \overline{u}) = F(f(t, x, u, a), g(t, x, u, a), h(t, x, u, a)) = F(t, x, u) \tag{1.14}$$

for $t, x, u, a \in \mathcal{D}' \subset \mathcal{D}$.

Theorem 1.3.2. A function $F(t, x, u)$ is an invariant of a group G with generator X if and only if $X(F) = 0$, i.e.,

$$X(F) = \xi^1\frac{\partial F}{\partial t} = \xi^2\frac{\partial F}{\partial x} = \eta\frac{\partial F}{\partial u}. \tag{1.15}$$

Condition (1.15) is known as the infinitesimal criterion of invariance. From (1.15), one-parameter group has 2 functionally independent invariants (basis of invariants) taken from the left hand sides of 2 first integrals $J_1(t, x, u) = C_1$ and $J_2(t, x, u) = C_2$ of the characteristic equations for linear PDE (1.15). Any other invariant is a function of $J_1(t, x, u) = C_1$ and $J_2(t, x, u) = C_2$.

1.4 Prolongation formulas

Consider a kth-order PDE

$$E(t, x, u, u_{(1)}, \ldots, u_{(k)}) = 0, \tag{1.16}$$

where t and x are two independent variables and u is the dependent variable. Let

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{1.17}$$

be the infinitesimal generator of the one-parameter group G of transformation (1.3).

Definition 1.4.1. The extended infinitesimal generator $X^{[k]}$ of the prolonged group $G^{[k]}$ on the space $(t, x, u, u_{(1)}, \ldots, u_{(k)})$ is called the kth prolongation of X, and denoted by

$$\begin{aligned} X^{[k]} = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \zeta_i(t,x,u,u_{(1)})\frac{\partial}{\partial u_{(1)}} \\ + \cdots + \zeta_{i_1\cdots i_k}(t,x,u,u_{(1)},\cdots,u_{(k)})\frac{\partial}{\partial u_{(i_1\cdots i_k)}}. \end{aligned} \tag{1.18}$$

The coefficients ζs are determined recursively by the prolongation formulae

$$\begin{aligned} \zeta_i &= D_i(\eta) - u_j D_i(\xi^j) \\ &= D_i(\eta) - u_t D_i(\xi^1) - u_x D_i(\xi^2), \\ \zeta_{ij} &= D_j(\zeta_i) - u_{il} D_j(\xi^l) \\ &= D_i(\zeta_i) - u_{it} D_j(\xi^1) - u_{ix} D_j(\xi^2), \\ &\vdots \\ \zeta_{i_1\cdots i_k} &= D_{i_k}(\zeta_{i_1\cdots i_{k-1}}) - u_{i_i\cdots i_{k-1}l} D_{i_k}(\xi^l) \\ &= D_{i_k}(\zeta_{i_1\cdots i_{k-1}}) - u_{i_i\cdots i_{k-1}t} D_{i_k}(\xi^1) - u_{i_i\cdots i_{k-1}x} D_{i_k}(\xi^2), \end{aligned} \tag{1.19}$$

where $i, j, i_1, \cdots, i_k = t, x$ and the total derivative operator D_i is given by

$$D_i = \frac{\partial}{\partial i} + u_j\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots . \tag{1.20}$$

For example, if we consider a second order partial differential equation (1.1), the second prolongation of the prolonged group $G^{[2]}$ in a space $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$ is given by

$$X^{[2]} = X + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{tt}\partial_{u_{tt}} + \zeta_{tx}\partial_{u_{tx}} + \zeta_{xx}\partial_{u_{xx}}. \tag{1.21}$$

The coefficients ζ_t, ζ_x, ζ_{tx} and ζ_{xx}, and are given by the formulae

$$\begin{aligned} \zeta_t &= D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \\ \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \\ \zeta_{tt} &= D_t(\zeta_t) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2), \\ \zeta_{tx} = \zeta_{xt} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \end{aligned}$$

where D_t and D_x are total derivatives with respect to t and x respectively given by

$$\begin{aligned} D_t =& \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{tt} \frac{\partial}{\partial u_t} + u_{tx} \frac{\partial}{\partial u_x} + \cdots , \\ D_x =& \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{tx} \frac{\partial}{\partial u_t} + u_{xx} \frac{\partial}{\partial u_x} + \cdots . \end{aligned} \tag{1.22}$$

Using the definitions of the total derivatives D_t and D_x yields

$$\zeta_t = \eta_t + u_t(\eta_u - \xi^1_t) - u_t^2 \xi^1_u - u_x \xi^2_t - u_x u_t \xi^2_u, \tag{1.23}$$

$$\zeta_x = \eta_x + u_x(\eta_u - \xi^2_x) - u_x^2 \xi^2_u - u_t \xi^1_x - u_x u_t \xi^1_u, \tag{1.24}$$

$$\begin{aligned} \zeta_{tt} = & \eta_{tt} + 2u_t \eta_{tu} + u_{tt}\eta_u + u_t^2 \eta_{uu} - 2u_{xt}\xi^2_t - u_x \xi^2_{tt} - 2u_x u_t \xi^2_{tu} - \xi^2_u(u_x u_{tt} + 2u_t u_{xt}) \\ & - u_x u_t^2 \xi^2_{uu} - 2u_{tt}\xi^1_t - u_t \xi^1_{tt} - 2u_t^2 \xi^1_{tu} - 3u_t u_{tt} \xi^1_u - u_t^3 \xi^1_{uu}, \end{aligned} \tag{1.25}$$

$$\begin{aligned} \zeta_{tx} = & \eta_{tx} + u_t \eta_{xu} + u_x \eta_{tu} + u_{xt}\eta_u + u_t u_x \eta_{uu} - u_{tt}\xi^1_x - u_{tx}\xi^1_t - \xi^1_u(u_x u_{tt} + 2u_t u_{tx}) \\ & - u_t(\xi^1_{xt} + u_t \xi^1_{xu} + u_x \xi^1_{tu} + u_t u_x \xi^1_{uu}) - u_{tx}\xi^2_x - u_{xx}\xi^2_t - \xi^2_u(u_t u_{xx} + 2u_x u_{xt}) \\ & - u_x(\xi^2_{tx} + u_t \xi^2_{xu} + u_x \xi^2_{tu} + u_t u_x \xi^2_{uu}), \end{aligned} \tag{1.26}$$

$$\begin{aligned} \zeta_{xx} = & \eta_{xx} + 2u_x \eta_{xu} + u_{xx}\eta_u + u_x^2 \eta_{uu} - 2u_{xx}\xi^2_x - u_x \xi^2_{xx} - 2u_x^2 \xi^2_{xu} - \xi^1_u(u_t u_{xx} + 2u_x u_{xt}) \\ & - u_x^3 \xi^2_{uu} - 2u_{xt}\xi^1_x - u_t \xi^1_{xx} - 2u_x u_t \xi^1_{xu} - u_x^2 u_t \xi^1_{uu} - 3u_x u_{xx} \xi^2_u. \end{aligned} \tag{1.27}$$

1.5 Determining equations for Lie point symmetries

In this section we introduce Lie's algorithm for calculating Lie point symmetries of differential equations.

Definition 1.5.1. An invertible transformation acting on the space (t, x, u) of E is a *point symmetry* of E provided it transform any solution of the equation into another solution of the same equation.

Theorem 1.5.1. Let G be a group of infinitesimal transformations (1.5) admitted by E. Then G consists of symmetries of E if and only if

$$X^{[2]}(E) = 0, \tag{1.28}$$

whenever (1.1) is satisfied for every group generator X of G. Conversely, the operator X is a point symmetry of (1.1) if

$$X^{[2]}(E) = 0, \quad \text{whenever } E = 0. \tag{1.29}$$

The symmetry condition (1.29), also called the invariance criterion can be written compactly as

$$X^{[2]}(E)\big|_{E=0} = 0, \tag{1.30}$$

where $|_{E=0}$ means evaluated at the equation $E = 0$.

Definition 1.5.2. The equation (1.30) is called the *determining equation*, it is an overdetermined system of linear homogeneous PDEs for the unknowns ξ^1, ξ^2, η (infinitesimals) of the symmetry generator X.

Theorem 1.5.2. The solution of the determining system form a vector space, that is, any finite linear combination of the symmetries is again a symmetry.

Theorem (1.5.1) summarizes Lie's algorithm. Lie's algorithm for calculating Lie point symmetries of differential equations is explained in several textbooks, among them [5, 7, 20, 21]. We will adopt Lie's algorithm as explained in [18].

Lie's algorithm

1. Write E such that all the terms are on the left hand side.

2. Write the symmetry generator
$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}.$$

3. Prolong the symmetry generator X to the order which is the same as the order of E, i.e.,
$$X^{[k]} = X + \zeta_i(t,x,u,u_{(1)})\frac{\partial}{\partial u_{(1)}} + \cdots + \zeta_{i_1\cdots i_k}(t,x,u,u_{(1)},\cdots,u_{(k)})\frac{\partial}{\partial u_{(i_1\cdots i_k)}}, \tag{1.31}$$
where the variables ζ_i are given by (1.19).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated at $E = 0$ which gives the symmetry condition
$$X^{[k]}(E)\big|_{E=0} = 0. \tag{1.32}$$

5. Substitute the ζ_i upon expansion of the symmetry condition and replace the derivatives which are to be eliminated.

6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting from an over-determined system of linear homogeneous PDEs in terms of infinitesimals ξ^1, ξ^2 and η.

7. Solve the over-determined system for the infinitesimals ξ^1, ξ^2 and η.

8. Construct the one-parameter group using Theorem (1.2.1).

1.6 Lie algebra

Definition 1.6.1. A Lie algebra is a vector space L over a field $\mathbb{F}$ with a binary operation $[-,-] : L\times L \to L$ called Lie bracket (also known as commutator), such that the following axioms are satisfied

(i) Bilinearity: If $X_1, X_2, X_3 \in L$ and $a, b \in \mathbb{F}$, then

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3].$$

(ii) Skew-Symmetry: If $X_1 \in L$, then

$$[X_1, X_1] = 0,$$

and this implies that, for all $X_1, X_2 \in L$

$$[X_1, X_2] = -[X_2, X_1].$$

(iii) Jacobi Identity: If $X_1, X_2, X_3 \in L$, then

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

Definition 1.6.2. A Lie bracket (commutator) $[-,-]$ on the set of vector fields $\mathcal{V}$ is defined by

$$[X_1, X_2] = X_1(X_2) - X_2(X_1),$$

where X_1 and X_2 are operators defined respectively by

$$X_1 = \xi_1^1(t, x, u)\frac{\partial}{\partial t} + \xi_1^2(t, x, u)\frac{\partial}{\partial x} + \eta_1(t, x, u)\frac{\partial}{\partial u}$$

and

$$X_2 = \xi_2^1(t, x, u)\frac{\partial}{\partial t} + \xi_2^2(t, x, u)\frac{\partial}{\partial x} + \eta_2(t, x, u)\frac{\partial}{\partial u}.$$

Theorem 1.6.1. The set of all symmetries forms a Lie algebra called a symmetry Lie algebra.

Definition 1.6.3. The dimension of a Lie algebra L is the dimension of the finite vector space L. Finite-dimensional Lie algebra of dimension r is denoted by L_r

Definition 1.6.4. Let L_r be a Lie algebra over field $\mathbb{F}$, then a linear subspace S of L ($S \subseteq L$) is a subalgebra of L if it is closed under the Lie bracket of Ł, that is $[X_1, X_2] \in S$ if $X_1, X_2 \in S$.

Definition 1.6.5. Let I be a linear subspace of a Lie algebra L. Then I is an ideal of L if $[X_1, X_2] \in I$ whenever $X_1 \in I$ and $X_2 \in L$.

Definition 1.6.6. A Lie algebra L is called abelian if the Lie bracket vanishes for all elements in L, that is, $[X_1, X_2] = 0$ for all $X_1, X_2 \in L$.

1.7 Invariant solution

Definition 1.7.1. A solution $u = F(t, x)$ of E is invariant under the one-parameter group of transformation (1.3) if

$$\overline{u} = F(\bar{t}, \overline{x}). \tag{1.33}$$

Definition 1.7.2. A function $u = F(t, x)$ is said to be an invariant solution under the one-parameter group of transformation (1.3) of E if and only if

$$X(u - F(t, x))\big|_{u=F(t,x)} = 0. \tag{1.34}$$

1.8 Conclusion

In this chapter Lie group analysis of PDEs was introduced, where notations, theorems and definitions that will be used throughout this work were presented. An algorithm that is used to calculate Lie point symmetries of PDEs due to Sophus Lie was summarized.

Chapter 2

Symmetries and Invariant Solutions of a pseudo-parabolic PDE: Model I

In this chapter we consider the pseudo-parabolic PDE (4)

$$u_t = (u^p u_x + u^q u_{tx})_x, \tag{2.1}$$

where p and q are real numbers. We start by considering a general case where $p \neq q$ and $p, q \neq 0$ and perform Lie symmetry analysis, i.e, we find Lie point symmetries, then derive optimal systems, and subsequently perform symmetry reductions and construct group invariant solutions. Thereafter we consider particular cases of (2.1) for different values of p and q which arises from the analysis of the general case.

2.1 Calculation of Lie point symmetries

According to Lie's algorithm, the vector field

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{2.2}$$

is a symmetry generator of (2.1) if and only if

$$X^{[3]}(u_t - (u^p u_x + u^q u_{tx})_x)|_{u_t=(u^p u_x + u^q u_{tx})_x} = 0, \tag{2.3}$$

where $X^{[3]}$ is the third prolongation of X given by

$$X^{[3]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tt} \frac{\partial}{\partial u_{tt}} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{ttt} \frac{\partial}{\partial u_{ttt}} + \zeta_{ttx} \frac{\partial}{\partial u_{ttx}} + \zeta_{txx} \frac{\partial}{\partial u_{txx}} + \zeta_{xxx} \frac{\partial}{\partial u_{xxx}}.$$

From (2.3), we have

$$\begin{aligned}\big(\zeta_t - \big(p(p-1)u^{p-2}u_x^2 + pu^{p-1}u_{xx} + q(q-1)u^{q-2}u_x u_{tx} + qu^{q-1}u_{txx}\big)\,\eta \\ -(2pu^{p-1}u_x + qu^{q-1}u_{tx})\zeta_x - qu^{q-1}u_x\zeta_{tx} - u^p\zeta_{xx} \\ -u^q\zeta_{txx}\big)\big|_{u_t=(u^p u_x + u^q u_{tx})_x} = 0,\end{aligned} \tag{2.4}$$

where the coefficients ζ_t, ζ_x, ζ_{tx}, ζ_{xx} are given respectively by equations (1.23), (1.24), (1.26), (1.27), and ζ_{txx} is given by

$$\zeta_{txx} = D_t(\zeta_{xx}) - u_{xxt}D_t(\xi^1) - u_{xxx}D_t(\xi^2). \tag{2.5}$$

Expansion of (2.5) gives

$$\begin{aligned}\zeta_{txx} = {} & u_{txx}\eta_u - 2u_{ttx}(u_x\xi^1_u + \xi^1_x) - 2u_{txx}(u_x\xi^2_u + \xi^2_x) + 2u_{tx}(u_x\eta_{uu} + \eta_{xu}) \\ & - u_{tt}(u_{xx}\xi^1_u + u_x\xi^1_{xu} + u_x(u_x\xi^1_{uu} + \xi^1_{xu}) + \xi^1_{xx}) \\ & - u_{tx}(u_{xx}\xi^2_u + u_x\xi^2_{xu} + u_x(u_x\xi^2_{uu} + \xi^2_{xu}) + \xi^2_{xx}) \\ & + u_t(u_{xx}\eta_{uu} + u_x\eta_{xuu} + u_x(u_x\eta_{uuu} + \eta_{xuu}) + \eta_{xxu}) - u_{txx}(u_t\xi^1_u + \xi^1_t) \\ & - u_{xxx}(u_t\xi^2_u + \xi^2_t) + u_{xx}\eta_{tu} - 2u_{tx}(u_{tx}\xi^1_u + u_t(u_x\xi^1_{uu} + \xi^1_{xu}) + u_x\xi^1_{tu} + \xi^1_{tx}) \\ & - 2u_{xx}(u_{tx}\xi^2_u + u_t(u_x\xi^2_{uu} + \xi^2_{xu}) + u_x\xi^2_{tu} + \xi^2_{tx}) + u_x\eta_{txu} + u_x(u_x\eta_{tuu} + \eta_{txu}) \\ & + \eta_{txx} - u_t(u_{txx}\xi^1_u + 2u_{tx}(u_x\xi^1_{uu} + \xi^1_{xu}) + u_t(u_{xx}\xi^1_{uu} + u_x\xi^1_{xuu} \\ & + u_x(u_x\xi^1_{uuu} + \xi^1_{xuu}) + \xi^1_{xxu}) + u_{xx}\xi^1_{tu} + u_x\xi^1_{txu} + u_x(u_x\xi^1_{tuu} + \xi^1_{txu}) + \xi^1_{txx}) \\ & - u_x(u_{txx}\xi^2_u + 2u_{tx}(u_x\xi^2_{uu} + \xi^2_{xu}) + u_t(u_{xx}\xi^2_{uu} + u_x\xi^2_{xuu} \\ & + u_x(u_x\xi^2_{uuu} + \xi^2_{xuu}) + \xi^2_{xxu}) + u_{xx}\xi^2_{tu} + u_x\xi^2_{txu} + u_x(u_x\xi^2_{tuu} + \xi^2_{txu}) + \xi^2_{txx}).\end{aligned} \tag{2.6}$$

Substitution of equations (1.23), (1.24), (1.26), (1.27), and (2.6) into (2.4) yields an overdetermined system of linear partial differential equations (*determining equations*) that can be solved for the coefficients ξ^1, ξ^2 and η since they are independent of the derivatives of the dependent variable u. Generating the determining equations manually is easy but it is a lengthy and tiring task. The method of Lie is algorithmic, as a result, symbolic manipulation software packages such as Mathematica [25], Maple, Maxima and

Reduce have been developed to perform this task. In this work we will use Mathematica software package *YaLie* [7]. Thus, the determining equations are

$$\xi^1_u = 0, \tag{2.7}$$

$$\xi^2_u = 0, \tag{2.8}$$

$$\eta_{uu} = 0, \tag{2.9}$$

$$\xi^1_x = 0, \tag{2.10}$$

$$\eta_{xu} = 0, \tag{2.11}$$

$$\xi^2_t = 0, \tag{2.12}$$

$$qu^{q-1}\eta_x - u^q\xi^2_{xx} = 0, \tag{2.13}$$

$$u^p\eta_{xx} - \eta_t + u^q\eta_{txx} = 0, \tag{2.14}$$

$$qu^{q-1}\eta - 2u^q\xi^2_x = 0, \tag{2.15}$$

$$(1-p)pu^{p-2}\eta - pu^{p-1}\eta_u + 2pu^{p-1}\xi^2_x - pu^{p-1}\xi^1_t - qu^{q-1}\eta_{tu} = 0, \tag{2.16}$$

$$pu^{p-1}\eta - 2u^p\xi^2_x + u^p\xi^1_t + u^q\eta_{tu} = 0, \tag{2.17}$$

$$(1-q)qu^{q-2}\eta - qu^{q-1}\eta_u + 2qu^{q-1}\xi^2_x = 0, \tag{2.18}$$

$$2pu^{p-1}\eta_x - u^p\xi^2_{xx} + qu^{q-1}\eta_{tx} = 0, \tag{2.19}$$

where the subscripts denote partial derivatives with respect to the indicated variable. From equations (2.7) and (2.10) we have

$$\xi^1 = a(t), \tag{2.20}$$

where $a(t)$ is an arbitrary function of t. Equations (2.8) and (2.12) give

$$\xi^2 = b(x), \tag{2.21}$$

where $b(x)$ is an arbitrary function of x. Substituting equation (2.21) into equation (2.15) we obtain

$$\eta = \frac{2ub'(x)}{q}. \tag{2.22}$$

Equation (2.9) is identically satisfied by (2.22). Substituting equations (2.20), (2.21), and (2.22) into equations (2.11), (2.13), (2.14), (2.16), (2.17), (2.18) and (2.19) reduce to the following equations

$$b''(x) = 0, \tag{2.23}$$

$$a'(t) + 2(p-q)b'(x) = 0. \tag{2.24}$$

From equation (2.23), we obtain

$$b(x) = k_1 x + k_2, \tag{2.25}$$

where k_1 and k_2 are constants of integration. Substituting (2.25) into equation (2.24) we obtain

$$a'(t) + 2(p-q)k_1 = 0. \tag{2.26}$$

From equation (2.26) we get

$$a(t) = \frac{2k_1(q-p)t}{q} + k_3, \tag{2.27}$$

where k_3 is a constant of integration. Thus,

$$\begin{aligned} \xi^1 &= \frac{2k_1(q-p)t}{q} + k_3, \\ \xi^2 &= k_1 x + k_2, \\ \eta &= \frac{2k_1 u}{q}. \end{aligned} \tag{2.28}$$

Therefore the Lie point symmetries of equation (2.1) are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{2(q-p)t}{q}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{2u}{q}\frac{\partial}{\partial u} \approx 2(q-p)t\frac{\partial}{\partial t} + qx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.29}$$

We now consider particular cases of equation (2.1) and obtain corresponding Lie point symmetries.

Subcase 2.1.1. $p = q = \gamma \neq 0, 1$.

When $p = q = \gamma \neq 0, 1$ equation (2.1) becomes

$$u_t = (u^\gamma u_x + u^\gamma u_{tx})_x \tag{2.30}$$

and the Lie point symmetries of (2.30) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \gamma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.31}$$

Subcase 2.1.2. $p \neq 0, 1$ and $q = 0$.

When $p \neq 0, 1$ and $q = 0$ equation (2.1) becomes

$$u_t = (u^p u_x + u_{tx})_x \tag{2.32}$$

and the Lie point symmetries of (2.32) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= -pt\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \end{aligned} \tag{2.33}$$

Subcase 2.1.3. $p = q = 0$.

When $p = q = 0$ equation (2.1) becomes

$$u_t = u_{xx} + u_{txx} \tag{2.34}$$

and the Lie point symmetries of (2.34) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= u\frac{\partial}{\partial u}, \\ X_c &= c(t, x)\frac{\partial}{\partial u}. \end{aligned} \tag{2.35}$$

The function $c(t, x)$ is a solution to equation (2.34).

Subcase 2.1.4. $p = 0$ and $q = 1$.

When $p = 0$ and $q = 1$ equation (2.1) becomes

$$u_t = (u_x + uu_{tx})_x \tag{2.36}$$

and the Lie point symmetries of (2.36) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.37}$$

Subcase 2.1.5. $p = 1$ and $q = 0$.

When $p = 1$ and $q = 0$ equation (2.1) becomes

$$u_t = (uu_x + u_{tx})_x \tag{2.38}$$

and the Lie point symmetries of (2.38) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= -t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}. \end{aligned} \tag{2.39}$$

Subcase 2.1.6. $p \neq 0, 1$ and $q = 1$.

When $p \neq 0, 1$ and $q = 1$ equation (2.1) becomes

$$u_t = (u^p u_x + uu_{tx})_x \tag{2.40}$$

and the Lie point symmetries of (2.40) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 2(1-p)t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.41}$$

Subcase 2.1.7. $p = q = 1$.

When $p = q = 1$ equation (2.1) becomes

$$u_t = (uu_x + uu_{tx})_x \tag{2.42}$$

and the Lie point symmetries of (2.42) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.43}$$

Subcase 2.1.8. $p = 0$ and $q \neq 0, 1$.

When $p = 0$ and $q \neq 0, 1$ equation (2.1) becomes

$$u_t = (u_x + u^q u_{tx})_x \tag{2.44}$$

and the Lie point symmetries of (2.44) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 2qt\frac{\partial}{\partial t} + qx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.45}$$

Subcase 2.1.9. $p = 1$ and $q \neq 0, 1$.

When $p = 1$ and $q \neq 0, 1$ equation (2.1) becomes

$$u_t = (uu_x + u^q u_{tx})_x \tag{2.46}$$

and the Lie point symmetries of (2.46) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 2(q-1)t\frac{\partial}{\partial t} + qx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{2.47}$$

Remark 2.1.1. Subcases 2.1.4, 2.1.6, 2.1.8 and 2.1.9 are recovered from the general case. Subcase 2.1.7 is recovered from subcase 2.1.1 whereas subcase 2.1.5 is obtained from subcase 2.1.2. Therefore, this amounts to considering the general case and subcases 2.1.1, 2.1.2, 2.1.3 because they have different symmetry group.

2.2 Optimal system of subalgebras

In this section, we construct one-dimensional optimal system for the Lie algebra with basis (2.29) as an illustration, the rest are presented without detailed calculations.

Given a Lie algebra admitted by a differential equation, any linear combination of those Lie point symmetries can be used to perform symmetry reduction, and hence construct an invariant solution of the equation. Since there may be an infinite number of such linear combinations, it may not be possible to list all of them. Rather than guessing or choosing a good combination, there is an effective and systematic way of classifying those combinations known as obtaining an optimal system, which is a collection of all non-unique linear combinations that represents all possible linear combinations.

More formally, an optimal system of subalgebras is a list of conjugacy inequivalent subalgebras in the Lie algebra, L, where any other subalgebra in L is conjugate to only one of the subalgebras in that list using adjoint representation [20]. Patera and Winternitz [22] defined optimal system of subalgebras as a collection of pairwise non-conjugate subalgebras where two subalgebras in the Lie algebra are conjugate (similar) if there is a transformation from group G which takes one subalgebra into the other.

There are three methods mostly used to construct an optimal system of subalgebras: Olver [20] suggested a method where the most general expression of the Lie algebra is simplified as much as possible by subjecting it to various adjoint trasformations. Patera and Winternitz [22] in their method, they first start by finding the subalgebras of Lie algebra of dimension $r \leq 4$, then classify the subalgebras of each such Lie algebra into conjugacy classes and finally present a representative of each class. Conjugacy in each case is considered under the group of inner automorphism, i.e., the Lie group obtained by the exponentiation of the adjoint representation of the considered Lie algebra. In case the Lie algebra is of dimension $r > 4$, it is decomposed into decomposable parts then find the subalgebras for each and then combining them again. Method suggested by Ovsiannikov [21] construct an optimal system by using a global matrix for the adjoint transformation. In this work the former is used.

The adjoint transformation is expressed as the following series which is given by Baker-Campbell-Hausdorf formula [22]:

$$\begin{aligned} Ad(e^{\mu X_i})X_j &= \sum_{k=0}^{\infty} \frac{\mu^k}{k!}(AdX_i)^k X_j \\ &= X_j - \mu[X_i, X_j] + \frac{\mu^2}{2!}[X_i, [X_i, X_j]] - \frac{\mu^3}{3!}[X_i, [X_i, [X_i, X_j]]] + \cdots \end{aligned} \tag{2.48}$$

where μ is a parameter, and $[X_i, X_j]$ is the Lie bracket of X_i and X_j given by

$$[X_i, X_j] = X_i(X_j) - X_j(X_i).$$

Before we can construct an optimal system, we will illustrate with examples how a Lie bracket $[X_i, X_j]$ of two symmetries X_i and X_j is obtained and how the adjoint representation $Ad(e^{\mu X_i})X_j$ of X_i and X_j is constructed.

Example 2.2.1. Consider two symmetries of (2.1), say X_1 and X_3, then a non-zero commutator $[X_1, X_3]$ is given by

$$\begin{aligned}
[X_1, X_3] &= X_1(X_3) - X_3(X_1) \\
&= \frac{\partial}{\partial t}\left(2(q-p)t\frac{\partial}{\partial t} + qx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\right) - \left(2(q-p)t\frac{\partial}{\partial t} + qx\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\right)\frac{\partial}{\partial t} \\
&= 2(q-p)\frac{\partial}{\partial t} \\
&= 2(q-p)X_1.
\end{aligned}$$

The full calculations of the commutators $[X_i, X_j]$ where $i, j = 1, 2, 3$ are summarised in Table 2.1. To compute the adjoint representation, say $Ad(e^{\mu X_i})X_j$ of X_i and X_j we

$\nearrow [X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$2(q-p)X_1$
X_2	0	0	qX_2
X_3	$-2(q-p)X_1$	$-qX_2$	0

Table 2.1: Table of commutators of (2.1)

will make use of Table 2.1 and equation (2.48). As an illustration we will consider two examples.

Example 2.2.2. Consider two symmetries X_1 and X_3 of (2.1), the adjoint representation $Ad(e^{\mu X_1})X_3$ of X_1 and X_3 is given by

$$\begin{aligned}
Ad(e^{\mu X_1})X_3 &= \sum_{k=0}^{\infty} \frac{\mu^k}{k!}(AdX_1)^k X_3 \\
&= X_3 - \mu[X_1, X_3] + \frac{\mu^2}{2!}[X_1, [X_1, X_3]] - \cdots \\
&= X_3 - (2(q-p)X_1) + \frac{\mu^2}{2!}[X_1, 2(q-p)X_1] - \cdots \\
&= X_3 - 2(q-p)\mu X_1 + \frac{\mu^2}{2!}(0) \\
&= X_3 - 2(q-p)\mu X_1.
\end{aligned}$$

Example 2.2.3. Consider two symmetries X_2 and X_3 of (2.1), the adjoint representation $Ad(e^{\mu X_2})X_3$ of X_2 and X_3 is given by

$$\begin{aligned} Ad(e^{\mu X_3})X_2 &= \sum_{k=0}^{\infty} \frac{\mu^k}{k!}(AdX_3)^k X_2 \\ &= X_2 - \mu[X_3, X_2] + \frac{\mu^2}{2!}[X_3, [X_3, X_2]] - \frac{\mu^3}{3!}[X_3, [X_3, [X_3, X_2]]] + \cdots \\ &= X_2 - \mu(-qX_2) + \frac{\mu^2}{2!}[X_3, -qX_2] - \frac{\mu^3}{3!}[x_3, [X_3, -qX_2]] + \cdots \\ &= X_2 - \mu(-qX_2) + \frac{\mu^2}{2!}(-q)^2 X_2 - \frac{\mu^3}{3!}(-q)^2[X_3, X_2] + \cdots \\ &= X_2 - \mu(-qX_2) + \frac{\mu^2}{2!}(-q)^2 X_2 - \frac{\mu^3}{3!}(-q)^3 X_2 + \cdots \\ &= (1 + \mu q + \frac{(\mu q)^2)}{2} + \frac{(\mu q)^3}{3!} + \cdots)X_2 \\ &= e^{\mu q} X_2. \end{aligned}$$

The full calculations of the adjoint representations $Ad(e^{\mu X_i})X_j$ where $i, j = 1, 2, 3$ are summarised in Table 2.2. According to the method of constructing one-dimensional

$Ad(e^{\mu X_i})X_j$	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - 2(q-p)\mu X_1$
X_2	X_1	X_2	$X_3 - q\mu X_2$
X_3	$e^{2(q-p)\mu} X_1$	$e^{q\mu} X_2$	X_3

Table 2.2: Table of Adjoint representations of (2.1)

optimal system suggested by Olver [20], we set up a non-zero vector field or operator

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 \tag{2.49}$$

with arbitrary coefficients a_1, a_2 and a_3. The task is to simplify as many of the coefficients a_i as possible by acting on it by $Ad(e^{\mu X_i})$ of a group generated by X_i, this process of adjoint action eliminates $a_j X_j$ where j can be equal to i. The process is repeated until no further simplification is possible.

Firstly, suppose that $a_3 \neq 0$ and set $a_3 = 1$ without loss of generality. Then the vector X becomes

$$X = a_1 X_1 + a_2 X_2 + X_3. \tag{2.50}$$

To eliminate the coefficient of X_1, we act on such a vector X by $Ad(e^{\frac{a_1}{2(q-p)}X_1})$, the vector becomes

$$\begin{aligned} X' &= Ad(e^{\frac{a_1}{2(q-p)}X_1})X \\ &= a_1X_1 + a_2X_2 + X_3 - \frac{a_1}{2(q-p)}2(q-p)X_1 \\ &= a_2X_2 + X_3. \end{aligned} \tag{2.51}$$

We continue to eliminate the coefficient of X_2 by acting on X' by $Ad(e^{\frac{a_2}{q}X_2})$, then the vector becomes

$$\begin{aligned} X'' &= Ad(e^{\frac{a_2}{q}X_2})X' \\ &= a_2X_2 + X_3 - \frac{a_2}{q}qX_2 \\ &= X_3. \end{aligned} \tag{2.52}$$

Therefore X is equivalent to X_3 under the adjoint representation, that is, every one-dimensional subalgebra generated by a vector X with $a_3 \neq 0$ is equivalent to a subalgebra spanned by X_3.

Secondly, suppose that $a_3 = 0$ and $a_1 \neq 0$. Scaling X if necessary we assume that $a_1 = 1$, then the vector X becomes

$$X = X_1 + a_2X_2. \tag{2.53}$$

We act on X by $Ad(e^{\mu X_3})$ a group generated by X_3 so that

$$\begin{aligned} X' &= Ad(e^{\mu X_3})X \\ &= e^{2\mu(q-p)}X_1 + a_2e^{\mu q}X_2. \end{aligned} \tag{2.54}$$

The vector X' is a scalar multiple of the vector $X'' = X_1 + a_2e^{2\mu p - \mu q}X_2$, hence depending on the sign of a_2 we can make the coefficient of X_2 either $+1$ or -1. Thus any one-dimensional subalgebra generated by X with $a_3 = 0$ and $a_1 \neq 0$ is equivalent to a subalgebra spanned by $X_1 + \delta X_2$ where $\delta = +1, -1$. No further simplification is possible.

Thirdly, suppose that $a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$. Scaling X we let $a_1 = 1$, then the vector X becomes

$$X = X_1 \tag{2.55}$$

which can not be simplified further. Thus any subalgebra generated by X with $a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$ is equivalent to a subalebra spanned by X_1. No further simplifications on X are possible.

Lastly, suppose that $a_1 = a_3 = 0$ and $a_2 \neq 0$. Scaling X we let $a_2 = 1$, then the vector X becomes

$$X = X_2 \tag{2.56}$$

which can not be simplified further. Thus any subalgebra generated by X with $a_1 = a_3 = 0$ and $a_2 \neq 0$ is equivalent to a subalebra spanned by X_2. No further simplifications on X are possible.

In summary, an optimal system of one-dimensional subalgebras is spanned by

$$\begin{aligned} & X_3, \\ & X_1 + \delta X_2; \qquad \delta = \pm 1, \\ & X_2, \\ & X_1. \end{aligned} \tag{2.57}$$

Optimal systems of different subcases of equation (2.1) considered in Section 2.1 are summarized in Table 2.3.

Subcase	Subalgebras	Conditions on constant(s)
2.1.1	$\{X_1, X_2, X_3, aX_1 + X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$, $a \neq 0$
2.1.2	$\{X_1, X_2, X_3, bX_2 + X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$, $b \neq 0$
2.1.3	{Linear combination of X_1, X_2, X_3}	
2.1.4	$\{X_1, X_2, X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$
2.1.5	$\{X_1, X_2, X_3, bX_2 + X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$, $b \neq 0$
2.1.6	$\{X_1, X_2, X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$
2.1.7	$\{X_1, X_2, X_3, X_1 + \delta X_2, aX_1 + X_3\}$	$\delta = \pm 1$, $a \neq 0$
2.1.8	$\{X_1, X_2, X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$
2.1.9	$\{X_1, X_2, X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$

Table 2.3: Optimal system of one-dimensional subalgeras of subcases of (2.1)

2.3 Symmetry reductions and invariant solutions

In this section, we perform all possible similarity (symmetry) reductions and construct invariant solutions using the optimal system obtained in Section 2.2. That is, a symmetry or linear combination of symmetries is utilized to reduce a PDE into an (ODE) and then an appropriate fundamental method is employed to solve the ODE. Below we give calculations for subcases 2.1.3, 2.1.7 and present the rest in a tabular form.

Subcase 2.1.3

We note that the symmetry Lie algebra is infinite dimensional and thus, only the finite part of the Lie algebra is considered. Invariance under one of X_1, X_2 and X_3 is trivial, hence we take any linear combination of X_1, X_2 and X_3.

(a) Invariance under $X_1 + X_2$. The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{1} = \frac{du}{0} \tag{2.58}$$

which give similarity variables as $J_1 = x - t$ and $J_2 = u$, therefore the invariant solution $J_2 = f(J_1)$ is

$$u(t,x) = f(x-t). \tag{2.59}$$

Substituting (2.59) into (2.34) we obtain the following ODE

$$f''' - f'' - f' = 0 \tag{2.60}$$

where "$'$" denotes differentiation with respect to $J_1 = x - t$, and it solves to

$$f(J_1) = \frac{\exp\left(\frac{(1-\sqrt{5})J_1}{2}\right)\left[-K_1(1+\sqrt{5}) + K_2(-1+\sqrt{5})\exp\left(\sqrt{5}J_1\right)\right]}{2} + K_3 \tag{2.61}$$

where K_1, K_2, K_3 are arbitrary constants. Finally the invariant solution becomes

$$u(t,x) = \frac{\exp\left(\frac{(1-\sqrt{5})(x-t)}{2}\right)\left[K_2(\sqrt{5}-1)\exp\left(\sqrt{5}(x-t)\right) - K_1(1+\sqrt{5})\right]}{2} + K_3. \tag{2.62}$$

(b) Invariance under $X_1 + X_3$. The characteristics equations are given by

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{u}. \tag{2.63}$$

Solving the characteristic equation give the similarity variables as $J_1 = x$ and $J_2 = ue^{-t}$. Therefore the invariant solution $J_2 = f(J_1)$ is given by

$$u(t,x) = e^{t} g(x). \tag{2.64}$$

Substituting (2.64) into (2.34) gives the following ODE

$$g - 2g'' = 0, \tag{2.65}$$

where "$'$" denotes differentiation with respect to the variable $J_1 = x$. The solution of equation (2.65) is

$$g(x) = K_4 \exp\left\{\sqrt{\frac{1}{2}}x\right\} + K_5 \exp\left\{-\sqrt{\frac{1}{2}}x\right\}, \tag{2.66}$$

where K_4 and K_5 are arbitrary constants. Hence the invariant solution is given by

$$u(t,x) = K_4 \exp\left\{t + \sqrt{\frac{1}{2}}x\right\} + K_5 \exp\left\{t - \sqrt{\frac{1}{2}}x\right\}. \tag{2.67}$$

(c) Invariance under $X_1 + X_2 - X_3$. The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{1} = \frac{du}{-u} \tag{2.68}$$

and yields the invariants $J_1 = x - t$ and $J_2 = ue^{t}$. Therefore the invariant solution is

$$u(t,x) = e^{-t} h(x-t). \tag{2.69}$$

Substituting (2.69) into (2.34) gives the following ODE

$$h + h' - h''' = 0, \tag{2.70}$$

where "$'$" denotes differentiation with respect to $J_1 = x - t$. Using Mathematica [25], the approximate roots to the auxilliary equation of (2.70) are $r_1 = 1.32472$ and $r_{2,3} = -0.662359 \pm 0.56228i$. Hence, we obtain

$$h(J_1) = K_6 e^{r_1 J_1} + e^{r_2 J_1}[K_7 \cos(r_3 J_1) + K_8 \sin(r_3 J_1)] \tag{2.71}$$

where K_6, K_7 and K_8 are arbitrary constants. Therefore, the invariant solution is given by

$$u(t,x) = e^{-t}\left[K_6 e^{(x-t)r_1} + e^{(x-t)r_2}\left(K_7 \cos\left[r_3(x-t)\right] + K_8 \sin\left[r_3(x-t)\right]\right)\right]. \tag{2.72}$$

(d) Invariance under $\alpha X_2 + X_3$; $\alpha \neq 0$. The characteristic equations are given by

$$\frac{dt}{0} = \frac{dx}{\alpha} = \frac{du}{u} \tag{2.73}$$

which give the invariants $J_1 = t$ and $J_2 = ue^{-x/\alpha}$. Therefore the invariant solution $J_2 = R(J_1)$ is given by

$$u(t,x) = e^{x/\alpha} R(t). \tag{2.74}$$

Substituting (2.74) into (2.34) gives the following ODE

$$\left(1-\alpha^2\right) R'(t) + R(t) = 0, \tag{2.75}$$

where "′" denotes differentiation with respect to the variable $J_1 = t$ and the solution is nonzero provided $\alpha \neq \pm 1$. The solution of equation (2.75) is

$$R(t) = K_9 \exp\left\{\frac{t}{\alpha^2-1}\right\} \tag{2.76}$$

where K_9 is an arbitrary constant. Hence the invariant solution is given by

$$u(t,x) = K_9 \exp\left\{\frac{x}{\alpha} + \frac{t}{\alpha^2-1}\right\}. \tag{2.77}$$

Subcase 2.1.7

Invariances under X_1 and X_2 are trivial hence not considered.

(a) Invariance under X_3. The characteristics equations are given by

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{2u} \tag{2.78}$$

which give invariants as $J_1 = t$ and $J_2 = u/x^2$. The invariant solution $J_2 = f(J_1)$ is given by

$$u(t,x) = x^2 f(t). \tag{2.79}$$

Substituting (2.79) into (2.42) we obtain the following ODE

$$f' - 6f^2 - 6ff' = 0, \tag{2.80}$$

where "′" denotes differentiation with respect to $J_1 = t$. The solution of equation (2.80) is given implicitly as

$$\frac{1}{6f(t)} + \ln f(t) = -x + K_9, \tag{2.81}$$

where K_9 is an arbitrary constant.

(b) Invariance under $X_1 + \delta X_2$. The characteristics equations are given by

$$\frac{dt}{1} = \frac{dx}{\delta} = \frac{du}{0} \tag{2.82}$$

which give the invariants as $J_1 = x - \delta t$ and $J_2 = u$, hence the invariant solution $J_2 = g(J_1)$ is given by

$$u(t,x) = g(x - \delta t). \tag{2.83}$$

Substituting (2.83) into (2.42) we obtain the following ODE

$$\delta g g''' + \delta g' g'' - g g'' - g'^2 - \delta g' = 0, \tag{2.84}$$

where "$'$" denotes differentiation with respect to $J_1 = x - \delta t$.

(c) Invariance under $aX_1 + X_3$. The characteristics equations are given by

$$\frac{dt}{a} = \frac{dx}{x} = \frac{du}{2u} \tag{2.85}$$

which give the invariants as $J_1 = xe^{-t/a}$ and $J_2 = ue^{-2t/a}$. Therefore the invariant solution is

$$u(t,x) = e^{2t/a} h(xe^{-t/a}). \tag{2.86}$$

Substituting (2.86) into (2.42) we obtain the following ODE

$$J_1 h' + (1+a)h'^2 - J_1 h' h'' - 2h + ahh'' - J_1 hh''' = 0, \tag{2.87}$$

where "$'$" denotes differentiation with respect to $J_1 = xe^{-t/a}$.

The group invariant solutions for the general case and other subcases are summarized in Table 2.4. The nonzero functions F, G, F_i, G_i and H_i of their respective arguments satisfy the ODEs

$$2F + 2p(p-q)F^{p-1}F'^2 + qF^{q-1}\left[(q-2)F'^2 + qzF'F''\right] + 2(p-q)F^pF''$$
$$-qzF' + F^q\left[2(q-1)F'' + qzF'''\right] = 0, \quad (2.88)$$
$$\left(pG'^2 + GG''\right)G^{p-1} - \delta G'\left(qG^{q-1}G'' - 1\right) - \delta G^qG''' = 0, \quad (2.89)$$
$$\gamma^2 F_1' - 2(2+\gamma)(F_1^{\gamma+1} + F_1^{\gamma}F_1') = 0, \quad (2.90)$$
$$cG_1' + G_1^{\gamma-1}[\gamma {G_1'}^2 - c\gamma G_1'G_1'' + G_1(G_1'' - cG_1''')] = 0, \quad (2.91)$$
$$(2H_1 - \gamma\kappa H_1')H_1^{1-\gamma} + \gamma(\gamma - a - 2){H_1'}^2 + \gamma^2\kappa H_1'H_1''$$
$$+H_1[(2\gamma - a - 2)H_1'' + \gamma\kappa H_1'''] = 0, \quad (2.92)$$
$$F_2^2 + p^2F_2^p{F_2'}^2 - F_2F_2'' + pF_2^{p+1}F_2'' = 0, \quad (2.93)$$
$$\delta G_2' + pG_2^{p-1}{G_2'}^2 + G_2^pG_2'' - \delta G_2''' = 0, \quad (2.94)$$
$$p\left(p{H_2'}^2 + H_2H_2''\right){H_2}^{p-1} + H_2 - bH_2' - H_2'' + bH_2''' = 0, \quad (2.95)$$
$${F_4'}^2 + \phi F_4'(1 - F_4'') + 2F_4'' - F_4(2 + \phi F_4''') = 0, \quad (2.96)$$
$$\delta G_4'(G_4'' - 1) - G_4'' + \delta G_4G_4''' = 0, \quad (2.97)$$
$$F_5 + {F_5'}^2 - F_5'' + F_5F_5'' = 0, \quad (2.98)$$
$$\delta G_5' + {G_5'}^2 + G_5G_5'' - \delta G_5''' = 0, \quad (2.99)$$
$$(b - H_5')H_5' + H_5'' - H_5(1 + H_5'') - bH_5''' = 0, \quad (2.100)$$
$$2(p-1)(p{F_6'}^2 + F_6F_6'')F_6^{p-1} + F_6'[\omega(F_6'' - 1) - F_6'] + F_6(2 + \omega F_6''') = 0, \quad (2.101)$$
$$\delta G_6'(G_6'' - 1) + \delta G_6G_6''' - G_6^{p-1}(p{G_6'}^2 + G_6G_6'') = 0, \quad (2.102)$$
$$[q\theta F_7' - 2(F_7 - qF_7'')]F_7^{1-q} - q(q-2){F_7'}^2 - q^2\theta F_7'F_7''$$
$$-F_7[2(q-1)F_7'' + q\theta F_7'''] = 0, \quad (2.103)$$
$$\delta G_7'\left(qG_7^{q-1}G_7'' - 1\right) - G_7'' + \delta G_7^qG_7''' = 0, \quad (2.104)$$
$$[F_8'(q\vartheta + 2[q-1]F_8') + 2F_8([q-1]F_8'' - 1)]F_8^{1-q} - q(q-2){F_8'}^2$$
$$-q^2\vartheta F_8'F_8'' - F_8[2(q-1)F_8'' + q\vartheta F_8'''] = 0, \quad (2.105)$$
$$G_8'\left(\delta - \delta qG_8^{q-1}G_8'' + G_8'\right) + G_8G_8'' - \delta G_8^qG_8''' = 0. \quad (2.106)$$

Equation	Invariant solution	Generated by sublagebra
(2.1)	$t^{1/(q-p)}F(z): \ z = t^{q/2(p-q)}x$	X_3
	$G(x - \delta t)$	$X_1 + \delta X_2$
(2.30)	$x^{2/\gamma}F_1(t)$	X_3
	$G_1(x - \delta t)$	$X_1 + \delta X_2$
	$e^{2t/a}H_1(\kappa): \ \kappa = xe^{-\gamma t/a}$	$aX_1 + X_3$
(2.32)	$t^{-1/p}F_2(x)$	X_3
	$G_2(x - \delta t)$	$X_1 + \delta X_2$
	$t^{-1/p}H_2(\varsigma): \ \varsigma = x + (b\ln t)/p$	$bX_2 + X_3$
(2.36)	$tF_4(\phi): \ \phi = t^{-1/2}x$	X_3
	$G_4(x - \delta t)$	$X_1 + \delta X_2$
(2.38)	$t^{-1}F_5(x)$	X_3
	$G_5(x - \delta t)$	$X_1 + \delta X_2$
	$t^{-1}H_5(\psi): \ x + b\ln t$	$bX_2 + X_3$
(2.40)	$t^{1/(1-p)}F_6(\omega): \ \omega = t^{-1/2(1-p)}x$	X_3
	$G_6(x - \delta t)$	$X_1 + \delta X_2$
(2.44)	$t^{1/q}F_7(\theta): \ \theta = t^{-1/2}x$	X_3
	$G_7(x - \delta t)$	$X_1 + \delta X_2$
(2.46)	$t^{1/(q-1)}F_8(\vartheta): \ \vartheta = t^{1/2(1-q)}x$	X_3
	$G_8(x - \delta t)$	$X_1 + \delta X_2$

Table 2.4: Group invariant solutions of (2.1) and subcases of (2.1)

2.4 Graphical Solutions

In this section, we present graphical solutions of group invariant solutions that were expressed explicitly in terms of independent variables t and x. Graphs of exact solutions of subcase 2.1.3 are presented below.

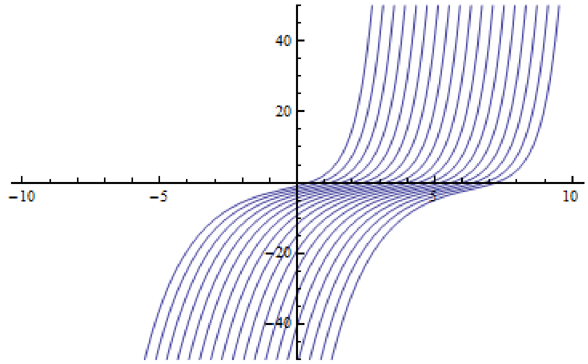

Figure 2.1: Solution (2.62) with $K_1 = K_2 = 1, K_3 = 0$.

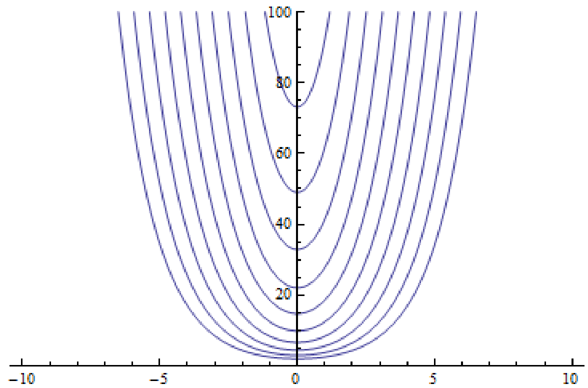

Figure 2.2: Solution (2.67) with $K_4 = K_5 = 1$.

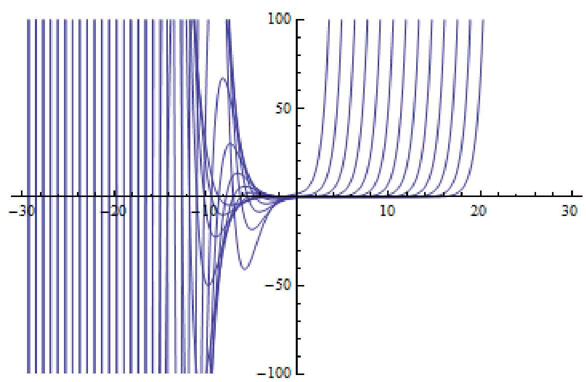

Figure 2.3: Solution (2.72) with $K_6 = K_7 = K_8 = 1$.

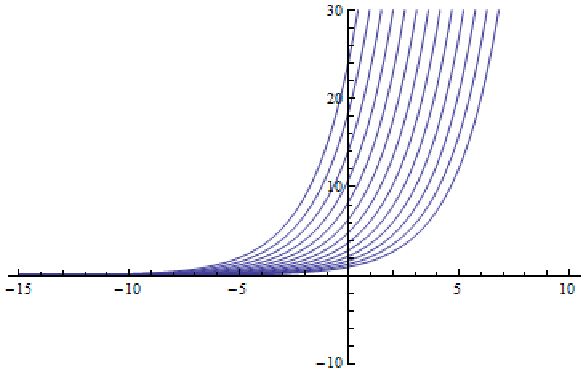

Figure 2.4: Solution (2.77) with $K_9 = 1, \alpha = 2$.

2.5 Conclusion

In this chapter we obtained the symmetry Lie algebra of the pseudo-parabolic PDE for power law in diffusion coefficient with constant viscosity, different subcases were considered. Optimal systems of one-dimensional subalgebras were then derived, and subsequently used to perform symmetry reductions and construct group invariant solutions. Graphical solutions were presented where group invariant solutions were expressed explicitly in terms of the independent variables.

Chapter 3

Symmetries and Invariant Solutions of a pseudo-parabolic PDE: Model II

In this chapter we consider the pseudo-parabolic PDE (5)

$$u_t = (e^{mu}(u + e^{-nu}u_t)_x)_x. \tag{3.1}$$

Firstly, we perform Lie symmetry analysis of a general of a general case $m \neq n$ for $m, n > 0$. Secondly, we consider particular cases of (3.1) for different values of m and n which arises from the analysis of the general case and other cases of interest. In each case the Lie point symmetries are obtained and then used to perform symmetry reductions and/or construct group invariant solutions.

3.1 Lie point symmetries

According to Lie's algorithm, the generator of Lie point symmetries of (3.1) is

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3.2}$$

if and only if

$$X^{[3]}(u_t - (e^{mu}(u + e^{-nu}u_t)_x)_x)|_{(3.1)} = 0, \tag{3.3}$$

where $X^{[3]}$ is the third prolongation of X given by

$$X^{[3]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{txx} \frac{\partial}{\partial u_{txx}}.$$

From (3.3), we have

$$\begin{aligned} \big(- e^{(m-n)u}\zeta_{txx} - (m-2n)e^{(m-n)u}u_x\zeta_{tx} + (1 + ne^{(m-n)u}((m-n)u_x^2 + u_{xx}))\zeta_t \\ -e^{mu}(1 - ne^{-nu}u_t)\zeta_{xx} - e^{(m-n)u}(2u_x(e^{nu}m + n(n-m)u_t) + (m-2n)u_{tx})\zeta_x \\ +e^{(m-n)u}(-e^{nu}m(mu_x^2 + u_{xx}) \\ +(n-m)(-n((m-n)u_x^2 + u_{xx})u_t + (m-2n)u_xu_{tx} + u_{txx}))\eta\big)\big|_{(3.1)} = 0, \end{aligned} \tag{3.4}$$

where the coefficients ζ_t, ζ_x, ζ_{tx}, ζ_{xx} are given respectively by equations (1.23), (1.24), (1.26), (1.27), and (2.6). Substitution of equations (1.23), (1.24), (1.26), (1.27), and (2.6) into (3.4) yields an overdetermined system of linear homogeneous partial differential equations (*determining equations*) that can be solved for the coefficients ξ^1, ξ^2 and η of the symmetry generator (3.2). Generating the determining equations manually is easy but tedious, often some terms are omitted by mistake and thus, lead to the wrong solution.

With the aid of *YaLie* [7] software package, the determining equations become

$$\xi^1_u = 0, \tag{3.5}$$

$$\xi^2_u = 0, \tag{3.6}$$

$$\xi^1_x = 0, \tag{3.7}$$

$$\xi^2_t = 0, \tag{3.8}$$

$$n\eta_u - \eta_{uu} = 0, \tag{3.9}$$

$$(2n-m)\eta_u - 2\eta_{uu} = 0, \tag{3.10}$$

$$(2mn - 2n^2)\eta_u + (3n-m)\eta_{uu} - \eta_{uuu} = 0, \tag{3.11}$$

$$(2n-m)\eta_x - 2\eta_{xu} + \xi^2_{xx} = 0, \tag{3.12}$$

$$n\eta - ne^{-nu}\eta_t + \xi^1_t + e^{-nu}\eta_{tu} = 0, \tag{3.13}$$

$$(2mn - 2n^2)\eta_x + (4n-m)\eta_{xu} - 2\eta_{xuu} - n\xi^2_{xx} = 0, \tag{3.14}$$

$$\begin{aligned} mn\eta + m\eta_u + \eta_{uu} - \left(mne^{-nu} - n^2e^{-nu}\right)\eta_t + m\xi^1_t - \left(2ne^{-nu} - me^{-nu}\right)\eta_{tu} \\ +e^{-nu}\eta_{tuu} = 0, \end{aligned} \tag{3.15}$$

$$\eta_{xx} - \eta_t + e^{-nu}\eta_{txx} = 0, \tag{3.16}$$

$$(n-m)\eta + 2\xi^2_x + ne^{(m-n)u}\eta_{xx} - e^{(m-n)u}\eta_{xxu} = 0, \tag{3.17}$$

$$2m\eta_x + 2\eta_{xu} - \xi^2_{xx} - \left(2ne^{-nu} - me^{-nu}\right)\eta_{tx} + 2e^{-nu}\eta_{txu} = 0, \tag{3.18}$$

where the subscripts denote partial derivatives with respect to the indicated variable(s).

From equations (3.5) and (3.7) we obtain

$$\xi^1 = \xi^1(t). \tag{3.19}$$

From equations (3.6) and (3.8) we obtain

$$\xi^2 = \xi^2(x). \tag{3.20}$$

From (3.9) and (3.10), we get

$$m\eta_u = 0$$

which implies

$$\eta = \eta(t,x), \quad \text{since } m \neq 0. \tag{3.21}$$

From (3.12), (3.13), (3.14) and (3.21), we obtain

$$\eta = l_1, \tag{3.22}$$

where l_1 is a constant of integration. From (3.12) and (3.22), we obtain

$$\xi^2(x) = l_2 x + l_3, \tag{3.23}$$

where l_2 and l_3 are constants of integration. Substituting (3.22) and (3.23) into (3.17) we obtain

$$l_1 = \frac{2}{m-n} l_2 = \eta. \tag{3.24}$$

Also, substituting equations (3.22), (3.23) and (3.24) into equations (3.13) and (3.15) we get

$$\xi^1(t) = \frac{2nl_2 t}{n-m} + l_3, \tag{3.25}$$

where l_3 is a constant of integration. The remaining equations are identically satisfied. Therefore the Lie point symmetries of (3.1) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{2nt}{n-m}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{2}{m-n}\frac{\partial}{\partial u} \approx 2nt\frac{\partial}{\partial t} + (n-m)x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}. \end{aligned} \tag{3.26}$$

We now consider particular cases of (3.1) and find corresponding Lie point symmetries.

Subcase 3.1.1. $m = n$.

When $m = n$ equation (3.1) becomes

$$u_t = (e^{mu}(u + e^{-mu}u_t)_x)_x \tag{3.27}$$

and the Lie point symmetries of (3.27) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= mt\frac{\partial}{\partial t} - \frac{\partial}{\partial u}. \end{aligned} \tag{3.28}$$

Subcase 3.1.2. $m = 0$ and $n \neq 0$.

When $m = 0$ and $n \neq 0$ equation (3.1) becomes

$$u_t = (u + e^{-nu}u_t)_{xx} \tag{3.29}$$

and the Lie point symmetries of (3.29) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= 2nt\frac{\partial}{\partial t} + nx\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}. \end{aligned} \tag{3.30}$$

Subcase 3.1.3. $m \neq 0$ and $n = 0$.

When $m \neq 0$ and $n = 0$ equation (3.1) becomes

$$u_t = (e^{mu}(u + u_t)_x)_x \tag{3.31}$$

and the Lie point symmetries of (3.31) are given by

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= mx\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}. \end{aligned} \tag{3.32}$$

Remark 3.1.1. Subcases 3.1.1, 3.1.2 and 3.1.3 are recovered from the general case.

3.2 Optimal system of subalgebras

All the cases in the last section have three-dimensional Lie algebra, any linear combination of the Lie point symmetries can be used to perform symmetry reduction, and hence construct an invariant solution of the equation. However, in order to obtain all possible linear combinations without guessing or a good combination, an optimal system of one-dimensional subalgebras is constructed. In constructing one-dimensional system [20], we illustrate with an example by considering the Lie algebra having the basis (3.26), the rest of the cases are presented without derivation.

Example 3.2.1. Consider two symmetries of (3.1), say X_1 and X_3, then a non-zero commutator $[X_1, X_3]$ is given by

$$\begin{aligned}
[X_1, X_3] &= X_1(X_3) - X_3(X_1) \\
&= \frac{\partial}{\partial t}\left(2nt\frac{\partial}{\partial t} + (n-m)x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}\right) - \left(2nt\frac{\partial}{\partial t} + (n-m)x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}\right)\frac{\partial}{\partial t} \\
&= 2n\frac{\partial}{\partial t} \\
&= 2nX_1.
\end{aligned}$$

The full calculations of the commutators $[X_i, X_j]$ where $i, j = 1, 2, 3$ are summarised in Table 3.1. To compute the adjoint representation, say $Ad(e^{\mu X_i})X_j$ of X_i and X_j we

$\nearrow [X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$2nX_1$
X_2	0	0	$(n-m)X_2$
X_3	$-2nX_1$	$-(n-m)X_2$	0

Table 3.1: Table of commutators of (3.1)

will make use of Table 3.1 and equation (2.48). As an illustration we will consider two examples.

Example 3.2.2. Consider two symmetries X_2 and X_3 of (3.1), the adjoint representation

$Ad(e^{\mu X_2})X_3$ of X_2 and X_3 is given by

$$\begin{aligned}
Ad(e^{\mu X_2})X_3 &= \sum_{k=0}^{\infty} \frac{\mu^k}{k!}(AdX_2)^k X_3 \\
&= X_3 - \mu[X_2, X_3] + \frac{\mu^2}{2!}[X_2, [X_2, X_3]] - \cdots \\
&= X_3 - \mu(n-m)X_2 + \frac{\mu^2}{2!}[X_2, (n-m)X_2] - \cdots \\
&= X_3 - \mu(n-m)X_2 + \frac{\mu^2}{2!}(0) \\
&= X_3 - \mu(n-m)X_2.
\end{aligned}$$

Example 3.2.3. Consider two symmetries X_3 and X_1 of (3.1), the adjoint representation $Ad(e^{\mu X_1})X_3$ of X_1 and X_3 is given by

$$\begin{aligned}
Ad(e^{\mu X_3})X_1 &= \sum_{k=0}^{\infty} \frac{\mu^k}{k!}(AdX_3)^k X_1 \\
&= X_1 - \mu[X_3, X_1] + \frac{\mu^2}{2!}[X_3, [X_3, X_1]] - \frac{\mu^3}{3!}[X_3, [X_3, [X_3, X_1]]] + \cdots \\
&= X_1 - \mu(-2nX_1) + \frac{\mu^2}{2!}[X_3, -2nX_1] - \frac{\mu^3}{3!}[x_3, [X_3, -2nX_1]] + \cdots \\
&= X_1 - \mu(-2nX_1) + \frac{\mu^2}{2!}(-2n)^2 X_1 - \frac{\mu^3}{3!}(-2n)^2[X_3, X_1] + \cdots \\
&= X_1 - \mu(-2nX_1) + \frac{\mu^2}{2!}(-2n)^2 X_1 - \frac{\mu^3}{3!}(-2n)^3 X_1 + \cdots \\
&= (1 + \mu 2n + \frac{(\mu 2n)^2}{2} + \frac{(\mu 2n)^3}{3!} + \cdots)X_1 \\
&= e^{\mu 2n} X_1.
\end{aligned}$$

The full calculations of the adjoint representations $Ad(e^{\mu X_i})X_j$ where $i, j = 1, 2, 3$ are summarised in Table 3.2. We set a non-zero vector field or operator

$Ad(e^{\mu X_i})X_j$	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - 2n\mu X_1$
X_2	X_1	X_2	$X_3 - (n-m)\mu X_2$
X_3	$e^{2n\mu}X_1$	$e^{(n-m)\mu}X_2$	X_3

Table 3.2: Table of Adjoint representations of (3.1)

$$X = a_1X_1 + a_2X_2 + a_3X_3 \tag{3.33}$$

with arbitrary coefficients a_1, a_2 and a_3. The task is to simplify as many of the coefficients a_i as possible by acting on it by $Ad(e^{\mu X_i})$ of a group generated by X_i, this process of adjoint action eliminates a_jX_j where j can be equal to i. The process is repeated until no further simplification is possible.

Firstly, suppose that $a_3 \neq 0$ and set $a_3 = 1$ without loss of generality. Then the vector X becomes

$$X = a_1X_1 + a_2X_2 + X_3. \tag{3.34}$$

To eliminate the coefficient of X_1, we act on such a vector X by $Ad(e^{\frac{a_1}{2n}X_1})$, the vector becomes

$$\begin{aligned} X' &= Ad(e^{\frac{a_1}{2n}X_1})X \\ &= a_1X_1 + a_2X_2 + X_3 - \frac{a_1}{2n}2nX_1 \\ &= a_2X_2 + X_3. \end{aligned} \tag{3.35}$$

We continue to eliminate the coefficient of X_2 by acting on X' by $Ad(e^{\frac{a_2}{n-m}X_2})$, then the vector becomes

$$\begin{aligned} X'' &= Ad(e^{\frac{a_2}{n-m}X_2})X' \\ &= a_2X_2 + X_3 - \frac{a_2}{n-m}(n-m)X_2 \\ &= X_3. \end{aligned} \tag{3.36}$$

Therefore X is equivalent to X_3 under the adjoint representation, that is, every one-dimensional subalgebra generated by a vector X with $a_3 \neq 0$ is equivalent to a subalgebra spanned by X_3.

Secondly, suppose that $a_3 = 0$ and $a_1 \neq 0$. Scaling X if necessary we assume that $a_1 = 1$, then the vector X becomes

$$X = X_1 + a_2X_2. \tag{3.37}$$

We act on X by $Ad(e^{\mu X_3})$ a group generated by X_3 so that

$$\begin{aligned} X' &= Ad(e^{\mu X_3})X \\ &= e^{2n\mu}X_1 + a_2e^{\mu(n-m)}X_2. \end{aligned} \tag{3.38}$$

The vector X' is a scalar multiple of the vector $X'' = X_1 + a_2 e^{-\mu(m+n)} X_2$, hence depending on the sign of a_2 we can make the coefficient of X_2 either $+1$ or -1. Thus any one-dimensional subalgebra generated by X with $a_3 = 0$ and $a_1 \neq 0$ is equivalent to a subalgebra spanned by $X_1 + \delta X_2$ where $\delta = +1, -1$. No further simplification is possible.

Thirdly, suppose that $a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$. Scaling X we let $a_1 = 1$, then the vector X becomes

$$X = X_1, \tag{3.39}$$

which can not be simplified further. Thus any subalgebra generated by X with $a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$ is equivalent to a subalebra spanned by X_1. No further simplifications on X are possible.

Lastly, suppose that $a_1 = a_3 = 0$ and $a_2 \neq 0$. Scaling X we let $a_2 = 1$, then the vector X becomes

$$X = X_2, \tag{3.40}$$

which can not be simplified further. Thus any subalgebra generated by X with $a_1 = a_3 = 0$ and $a_2 \neq 0$ is equivalent to a subalebra spanned by X_2. No further simplifications on X are possible.

In summary, an optimal system of one-dimensional subalgebras is spanned by

$$\begin{aligned} &X_3, \\ &X_1 + \delta X_2, \qquad \delta = \pm 1, \\ &X_2, \\ &X_1. \end{aligned} \tag{3.41}$$

Optimal systems of different subcases of equation (3.1) considered in Section 3.1 are summarized in Table 3.3.

3.3 Symmetry reductions and invariant solutions

In this section, we perform all possible similarity (symmetry) reductions and construct invariant solutions for subcase 3.1.3 using the optimal system obtained in Section 3.2.

Subcase	Subalgebras	Conditions on constant(s)
3.1.1	$\{X_1, X_2, X_3, aX_2 + X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$, $a \neq 0$
3.1.2	$\{X_1, X_2, X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$
3.1.3	$\{X_1, X_2, X_3, aX_1 + X_3, X_1 + \delta X_2\}$	$\delta = \pm 1$, $a \neq 0$

Table 3.3: Optimal system of one-dimensional subalgeras of subcases of (3.1)

Subcase 3.1.3

Invariance under X_1 and X_2 are trivial hence not considered.

(a) Invariance under X_3. The characteristics equations are given by

$$\frac{dt}{0} = \frac{dx}{mx} = \frac{du}{2}. \tag{3.42}$$

Solving the characteristic equation give the similarity variables as $J_1 = t$ and $J_2 = u - \ln(x^2)/m$, therefore the invariant solution $J_2 = f(J_1)$ is given by

$$u(t, x) = \frac{\ln x^2}{m} + f(t). \tag{3.43}$$

Substituting (3.43) into (3.31) gives the following ODE

$$mf' - 2e^{mf} = 0, \tag{3.44}$$

where "$'$" denotes differentiation with respect to the variable $J_1 = t$ and it solves to

$$f(t) = -\frac{\ln(C - 2t)}{m} \tag{3.45}$$

where C is a constant of integration. Therefore the invariant solution becomes

$$u(t, x) = \ln\left(\frac{x^2}{C - 2t}\right)^{1/m}. \tag{3.46}$$

(b) Invariance under $X_1 + \delta X_2$. The characteristic equations are given by

$$\frac{dt}{1} = \frac{dx}{\delta} = \frac{du}{0} \tag{3.47}$$

which give similarity variables as $J_1 = x - \delta t$ and $J_2 = u$, therefore the invariant solution $J_2 = g(J_1)$ is

$$u(t, x) = g(x - \delta t). \tag{3.48}$$

Substituting (3.48) into (3.31) we obtain the following ODE

$$e^{mg_3}(\delta g_3''' - g_3'') - g_3'(\delta + me^{mg_3}(g_3' - \delta g_3'')) = 0, \tag{3.49}$$

where "′" denotes differentiation with respect to $J_1 = x - t$.

(c) Invariance under $aX_1 + X_3$. The characteristic equations are given by

$$\frac{dt}{a} = \frac{dx}{mx} = \frac{du}{2} \tag{3.50}$$

which give the invariants $J_1 = e^{-mt/a}x$ and $J_2 = u - 2t/a$, therefore the invariant solution $J_2 = h(J_1)$ is

$$u(t,x) = \frac{2t}{a} + h(e^{-mt/a}x). \tag{3.51}$$

Substituting (3.51) into (3.31) we obtain the following ODE

$$2 + m(m-a)e^{mh_3}{h_3'}^2 + (2m-a)e^{mh_3}h_3'' + m\psi h_3'(me^{mh_3}h_3'' - 1) + m\psi e^{mh_3}h_3''' = 0 \tag{3.52}$$

where "′" is the derivative with respect to $J_1 = e^{-mt/a}x$.

The group invariant solutions for the general case and other subcases are summarized in Table 3.4. The nonzero functions $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_i$, $\mathcal{G}_i$ and $\mathcal{H}_i$ of their respective arguments satisfy the ODEs

$$2e^{n\mathcal{F}} + e^{m\mathcal{F}}[m(m-n+2ne^{n\mathcal{F}})\mathcal{F}'^2 + 2(m+ne^{n\mathcal{F}})\mathcal{F}''] (n-m)z[e^{n\mathcal{F}}\mathcal{F}' - e^{m\mathcal{F}}(n(n-m)\mathcal{F}'^3 + (m-3n)\mathcal{F}'\mathcal{F}'' + \mathcal{F}''')] = 0, \tag{3.53}$$

$$\delta e^{n\mathcal{G}}\mathcal{G}' + e^{(m+n)\mathcal{G}}(m\mathcal{G}'^2 + \mathcal{G}'') + \delta e^{m\mathcal{G}}[n(m-n)\mathcal{G}'^3 + (m-3n)\mathcal{G}'\mathcal{G}'' + \mathcal{G}'''] = 0, \tag{3.54}$$

$$1 + m^2 e^{m\mathcal{F}_1}{\mathcal{F}_1'}^2 + m(1+e^{m\mathcal{F}_1})\mathcal{F}_1'' = 0, \tag{3.55}$$

$$me^{m\mathcal{G}_1}{\mathcal{G}_1'}^2 + e^{m\mathcal{G}_1}\mathcal{G}_1'' + \delta(1+2m\mathcal{G}_1'')\mathcal{G}_1' - \delta\mathcal{G}_1''' = 0, \tag{3.56}$$

$$1 - m\kappa^2(2a - e^{m\mathcal{H}_1}){\mathcal{H}_1'}^2 + \kappa^2(3a + m + me^{m\mathcal{H}_1})\mathcal{H}_1'' + a\kappa^3\mathcal{H}_1''' + m\mathcal{H}_1'(\kappa(1+e^{m\mathcal{H}_1}) - 2a\kappa^2\mathcal{H}_1'') = 0, \tag{3.57}$$

$$2e^{n\mathcal{F}_2} - n^3\varsigma\mathcal{F}_2' + 2ne^{n\mathcal{F}_2}\mathcal{F}_2'' + n\varsigma\mathcal{F}_2'(e^{n\mathcal{F}_2} + 3n\mathcal{F}_2'') - n\varsigma\mathcal{F}_2''' = 0, \tag{3.58}$$

$$\delta n^2{\mathcal{G}_2'}^3 - e^{n\mathcal{G}_2}\mathcal{G}_2'' - \delta\mathcal{G}_2'(e^{n\mathcal{G}_2} + 3n\mathcal{G}_2'') + \delta\mathcal{G}_2''' = 0. \tag{3.59}$$

Equation	Invariant solution	Generated by sublagebra
(3.1)	$-\frac{\ln t}{n} + \mathcal{F}(z):\ z = t^{(m-n)/2n}x$	X_3
	$\mathcal{G}(x - \delta t)$	$X_1 + \delta X_2$
(3.27)	$-\frac{\ln t}{m} + \mathcal{F}_1(x)$	X_3
	$\mathcal{G}_1(x - \delta t)$	$X_1 + \delta X_2$
	$-\frac{\ln t}{m} + \mathcal{H}_1(\kappa):\ \kappa = t^{-a/m}e^x$	$aX_2 + X_3$
(3.29)	$-\frac{\ln t}{n} + \mathcal{F}_2(\varsigma):\ \varsigma = t^{-1/2}x$	X_3
	$\mathcal{G}_2(x - \delta t)$	$X_1 + \delta X_2$

Table 3.4: Group invariant solutions of (3.1) and subcases of (3.1)

3.4 Graphical solutions

The graphical solution of group invariant solution of subcase 3.1.3 is presented below.

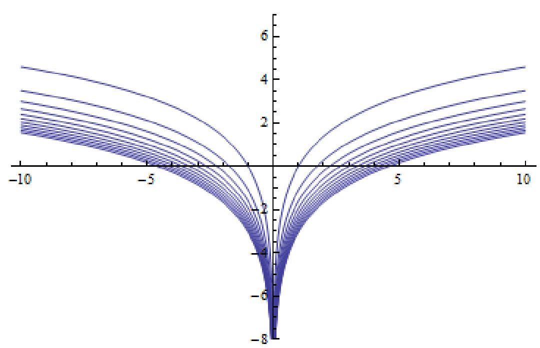

Figure 3.1: Solution (3.46) with $C = 21, m = 1$.

3.5 Conclusion

In this chapter we obtained the symmetry Lie algebra of the pseudo-parabolic PDE for exponential law in diffusion coefficient and viscosity, different subcases were considered. Optimal systems of one-dimensional subalgebras were then derived, and subsequently used to perform symmetry reductions and construct group invariant solutions. The graphical solution was presented.

Chapter 4

Approximate symmetry analysis of a perturbed pseudo-parabolic PDE: Model III

Partial differential equations with small parameter are widely used as mathematical models to describe non linear phenomena in various fields of mathematics, engineering and physical sciences such as mechanics, optics, e.t.c. For these perturbed equations, investigation of the analytic solutions play an important role in these related fields. Various perturbation methods have been developed to solve these equations, such as homotopy perturbation method, Adomian decomposition method, inverse scattering transformation method, e.t.c. [3].

In the 1980s Baikov, Gazizov and Ibragimov [2, 3] developed a method called approximate symmetry method, which is the combination of Lie group theory and perturbation analysis. In this method, the Lie operator is expanded in a perturbation series so that an approximate operator can be found. Also there is another method developed by Fushchich and Shtelen [11], where the dependent variables are expanded in a perturbation series; equations are separated at each order of approximation and the approximate symmetries of the original equations are defined to be the exact symmetries of the system obtained from equating to zero the coefficients of the smallness parameter.

In this work, the method due to Baikov et al. will be used to find the approximate symmetries. The definitions, theorems, proofs and notations of the theory of approximate symmetries in the first-order precision are based upon the references [16, 17].

4.1 Preliminaries

4.1.1 Notations and definitions

Consider a set of smooth vector functions depending on vector x and a group parameter a:

$$f_0(x,a), f_1(x,a)$$

with coordinates

$$f_0^i(x,a), f_1^i(x,a), \quad i = 1, \dots, n.$$

Let us define the one-parameter family G of approximate transformations

$$\bar{x}^i \approx f_0^i(x,a) + f_1^i(x,a), \quad i = 1, \dots, n. \tag{4.1}$$

of points $x = (x^1, \dots, x^n) \in \mathbb{R}^n$ into points $\bar{x} = (\bar{x}^1, \dots, \bar{x}^n) \in \mathbb{R}^n$ as the class of the invertible transformations

$$\bar{x} = f(x,a,\epsilon) \tag{4.2}$$

with vector-functions $f = (f^1, \dots, f^n)$ such that

$$f^i(x,a,\epsilon) \approx f_0^i(x,a) + f_1^i(x,a), \quad i = 1, \dots, n.$$

Here a is a real parameter, and the following condition is imposed:

$$f(x,0,\epsilon) \approx x.$$

Furthermore, it is assumed that the transformation (4.2) is defined for any value of a from a small neighborhood of $a = 0$, and that, in this neighborhood, the equation $f(x,a,\epsilon) \approx x$ yields $a = 0$.

Definition 4.1.1. The set of transformations (4.1) is called a one-parameter approximate transformation group if

$$f(f(x,a,\epsilon),b,\epsilon) \approx f(x,a+b,\epsilon)$$

for all transformations (4.2).

Remark 4.1.1. Here, unlike the usual classical Lie group theory, f does not necessarily denote the same function at each occurrence. It can be replaced by any function $g \approx f$.

Definition 4.1.2. The generator of an approximate transformation group G given by (4.2) is the class of first-order linear differential operators

$$X = \xi^i(x,\epsilon)\frac{\partial}{\partial x^i} \tag{4.3}$$

such that

$$\xi^i(x,\epsilon) \approx \xi_0^i(x) + \epsilon\xi_1^i(x),$$

where the vector fields ξ_0, ξ_1 are given by

$$\xi_0^i = \left.\frac{\partial f_0^i(x,a)}{\partial a}\right|_{a=0}, \quad \xi_1^i = \left.\frac{\partial f_1^i(x,a)}{\partial a}\right|_{a=0}, \quad i = 1,\dots,n.$$

In what follows, an approximate group generator

$$X \approx (\xi_0^i(x) + \epsilon\xi_1^i(x))\frac{\partial}{\partial x^i}$$

is written simply as

$$X = (\xi_0^i(x) + \epsilon\xi_1^i(x))\frac{\partial}{\partial x^i}. \tag{4.4}$$

Approximate Lie equations

Consider the one-parameter approximate transformation groups (4.1). Let

$$X = X_0 + \epsilon X_1 \tag{4.5}$$

be a given approximate operator where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}.$$

The corresponding approximate transformation (4.1) of points x into points $\bar{x} = \bar{x}_0 + \epsilon\bar{x}_1$ with the coordinates

$$\bar{x}^i = \bar{x}_0^i + \epsilon\bar{x}_1^i, \tag{4.6}$$

where

$$\bar{x}_0^i = f_0^i(x, a), \quad \bar{x}_1^i = f_0^i(x, a),$$

is determined by the following equations:

$$\frac{\mathrm{d}\bar{x}_0^i}{\mathrm{d}a} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i\big|_{a=0} = x^i, \quad i = 1, \dots, n, \tag{4.7}$$

$$\frac{\mathrm{d}\bar{x}_1^i}{\mathrm{d}a} = \sum_{k=1}^{n} \frac{\partial \xi_0^i(x)}{\partial x^k}\bigg|_{x=\bar{x}_0} \bar{x}_1^k + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i\big|_{a=0} = 0. \tag{4.8}$$

Equations (4.7) and (4.8) are called the *approximate Lie equations.*

Approximate exponential map

Theorem 4.1.1. Given the operator

$$X = X_0 + \epsilon X_1 \tag{4.9}$$

with a small parameter ϵ, where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \quad X_1 = \xi_1^i(x)\frac{\partial.}{\partial x^i}, \tag{4.10}$$

the corresponding approximate group transformation

$$\bar{x}^i = \bar{x}_0^i + \epsilon \bar{x}_1^i, \quad i = 1, \dots, n, \tag{4.11}$$

are determined by the following equations:

$$\bar{x}_0^i = e^{aX_0}(x^i), \quad \bar{x}_1^i = \langle\langle aX_0, aX_1\rangle\rangle(\bar{x}_0^i), \quad i = 1, \dots, n, \tag{4.12}$$

where

$$e^{aX_0} = 1 + aX_0 + \frac{a^2}{2!}X_0^2 + \frac{a^3}{3!}X_0^3 + \cdots \tag{4.13}$$

and

$$\langle\langle aX_0, aX_1\rangle\rangle = aX_1 + \frac{a^2}{2!}[X_0, X_1] + \frac{a^3}{3!}[X_0, [X_0, X_1]] + \cdots. \tag{4.14}$$

In other words, the approximate operator $X = X_0 + \epsilon X_1$ generates the one-parameter approximate transformation group given by the following approximate exponential map:

$$\bar{x}^i = (1 + \epsilon\langle\langle aX_0, aX_1\rangle\rangle)e^{aX_0}(x^i), \quad i = 1, \dots, n. \tag{4.15}$$

4.1.2 Approximate symmetries

Definition 4.1.3. Let G be a one-parameter approximate transformation group:

$$\bar{z}^i \approx f(z,a,\epsilon) \equiv f_0^i(z,a) + \epsilon f_1^i(z,a), \quad i = 1,\dots,n. \tag{4.16}$$

An approximate equation

$$F(z,\epsilon) \equiv F_0(z) + \epsilon F_1(z) \approx 0 \tag{4.17}$$

is said to be *approximately invariant* with respect to G, or *admits* G if

$$F(\bar{z},\epsilon) \approx F(f(z,a,\epsilon),\epsilon) = o(\epsilon) \tag{4.18}$$

whenever $z = (z^1,\dots,z^n)$ satisfies equation (4.17). If $z = (x,u,u_{(1)},\dots,u_{(k)})$, then (4.17) becomes an approximate differential equation of order k, and G is an *approximate symmetry group* of the differential equation.

Theorem 4.1.2. Equation (4.17) is approximately invariant under the approximate transformation group (4.16) with generator

$$X = X_0 + \epsilon X_1 = \xi_0^i(z)\frac{\partial}{\partial z^i} + \epsilon \xi_1^i(z)\frac{\partial}{\partial z^i}, \tag{4.19}$$

if and only if

$$\left[X^{[k]}F(z,\epsilon)\right]\Big|_{F\approx 0} = o(\epsilon) \tag{4.20}$$

or equivalently

$$\left[X_0^{[k]}F_0(z) + \epsilon\left(X_1^{[k]}F_0(z) + X_0^{[k]}F_1(z)\right)\right]\Big|_{(4.17)} = o(\epsilon) \tag{4.21}$$

in which k is order of equation and $X^{[k]}$ is k^{th}-order prolongation of X. The operator (4.19) satisfying equation (4.21) is called an *infinitesimal approximate symmetry* of, or an *approximate operator admitted* by (4.17). Accordingly, equation (4.21) is termed the *determining equation* for approximate symmetries.

Remark 4.1.2. The determining equation (4.21) can be written as follows:

$$X_0^{[k]}F_0(z) = \lambda(z)F_0(z), \tag{4.22}$$

$$X_1^{[k]}F_0(z) + X_0^{[k]}F_1(z) = \lambda(z)F_1(z). \tag{4.23}$$

The factor $\lambda(z)$ is determined by equation (4.22) and then substituted in equation (4.23). The latter equation must hold for all solutions of $F_0(z)$. Comparing equation (4.22) with the determining equation of exact symmetries, we obtain the following statement.

Theorem 4.1.3. If equation (4.17) admits an approximate transformation group with the generator $X = X_0 + \epsilon X_1$, where $X_0 \neq 0$, then the operator

$$X_0 = \xi_0^i \frac{\partial}{\partial z^i} \tag{4.24}$$

is an exact symmetry of the equation

$$F_0(z) = 0. \tag{4.25}$$

Remark 4.1.3. It is manifest from equation (4.22) and (4.23) that if X_0 is an exact symmetry of equation (4.25), then $X = \epsilon X_0$ is an approximate symmetry of (4.17).

Definition 4.1.4. Equations (4.25) and (4.17) are termed an *unperturbed equation* and a *perturbed equation* respectively. Under the conditions of Theorem (4.1.3), the operator X_0 is called a *stable symmetry* of the unperturbed equation (4.25). The corresponding approximate symmetry generator $X = X_0 + \epsilon X_1$ for the perturbed equation (4.17) is called a *deformation of the infinitesimal symmetry* X_0 of equation (4.25) caused by the perturbation $\epsilon F_1(z)$. In particular, if the most general symmetry Lie algebras of equation (4.25) is stable, we say that the perturbed equation (4.17) *inherits the symmetries of the unperturbed equation.*

Algorithm for calculating approximate symmetries

Remark (4.1.2) and Theorem (4.1.3) provide a simple and convenient algorithm for calculation of the first-order approximate symmetries of equations with a small parameter. The algorithm consists of the following three steps:

First step: Calculation of the exact symmetries X_0 of the unperturbed equation (4.25), e.g., by solving the determining equation

$$X_0^{[k]} F_0(z)\big|_{F_0(z)=0} = 0. \tag{4.26}$$

Second step: Determination of the auxiliary function H by virtue of equations (4.22), (4.23) and (4.17), i.e., by the equation

$$H = \frac{1}{\epsilon}\left[X_0^{[k]}(F_0(z) + \epsilon F_1(z))\big|_{F_0(z)+\epsilon F_1(z)=0}\right] \tag{4.27}$$

with known X_0 and $F_1(z)$.

Third step: Calculation of the operators X_1 by solving the determining equation for deformations:

$$X_1^{[k]} F_0(z)\big|_{F_0(z)=0} + H = 0. \tag{4.28}$$

Note that equation (4.28), unlike the determining equation (4.26) for exact symmetries is *inhomogeneous* and the prolongation formulas are the same as in the classical Lie theory.

Approximate Lie algebras

Definition 4.1.5. A class of first-order differential operator

$$X = \xi^i(z, \epsilon)\frac{\partial}{\partial z^i} \tag{4.29}$$

such that

$$\xi^i(z, \epsilon) \approx \xi_0^i(z) + \epsilon\xi_1^i(z), \quad i = 1, \ldots, n \tag{4.30}$$

with some fixed functions $\xi_0^i(z)$, $\xi_1^i(z)$, $i = 1, \ldots, n$, is called *approximate operator.*

Definition 4.1.6. An approximate commutator of the approximate operators X_1 and X_2 is an approximate operator denoted by $[X_1, X_2]$ and is given by

$$[X_1, X_2] \approx X_1(X_2) - X_2(X_1). \tag{4.31}$$

The approximate commutator satisfies the usual properties, namely

i (bi)linearity: $[aX_1 + bX_2, X_3] \approx a[X_1, X_3] + b[X_2, X_3], \quad a, b =$constants,

ii skew-symmetry: $[X_1, X_2] \approx -[X_2, X_1]$,

iii Jacobi identity: $[[X_1, X_2], X_3] + [[X_2, X_3], X_1] + [[X_3, X_1], X_2] \approx 0$.

Definition 4.1.7. A vector space L of approximate operators is called *an approximate Lie algebra of operators* if it is closed under the approximate commutator, i.e., if $[X_1, X_2] \in L$ for any $X_1, X_2 \in L$.

Remark 4.1.4. Here the approximate commutator $[X_1, X_2]$ is calculated to the first-order of precision.

Theorem 4.1.4. A set of approximate symmetries of an equation forms an approximate Lie algebra.

Approximate invariants

Definition 4.1.8. An approximate function $I(z, \epsilon)$ is called an approximate invariant of the group G of transformations (4.16) if for each $z \in \mathrm{R}^n$ and an admissible $a \in \mathrm{R}$

$$I(\bar{z}, \epsilon) \approx I(z, \epsilon). \tag{4.32}$$

Theorem 4.1.5. The approximate functionn is an approximate of the group G with the generator (4.29) if and only if the approximate equation

$$XF(z, \epsilon) \approx 0 \tag{4.33}$$

hold.

Theorem 4.1.6. Any one-parameter approximate group G with the generator (4.29) has exactly $n - 1$ functionally independent approximate invariants of the form

$$I^k(z, \epsilon) \approx I_0^k(z) + \epsilon I_1^k(z), \quad k = 1, \ldots, n-1, \tag{4.34}$$

and any approximate invariant of G can be represented in the form

$$I(z, \epsilon) = \varphi_0(I^1, \ldots, I^{n-1}) + \epsilon\varphi_1(I^1, \ldots, I^{n-1}), \tag{4.35}$$

where φ_0, φ_1 are arbitrary functions.

Approximately invariant solutions

Approximate invariants for the operator (4.19) are written in the form

$$J(z, \epsilon) = J_0(z) + \epsilon J_1(z) + o(\epsilon) \tag{4.36}$$

and are determined by the equation

$$X(J) = o(\epsilon), \tag{4.37}$$

or equivalently

$$X_0(J_0) + \epsilon(X_1(J_0) + X_0(J_1)) = 0. \tag{4.38}$$

This equation splits into the system:

$$X_0(J_0) = 0, \quad X_1(J_0) + X_0(J_1) = 0. \tag{4.39}$$

Solving equations (4.39) we will find two functionally independent invariants

$$\begin{aligned} J^1 &= J_0^1(z) + \epsilon J_1^1(z), \\ J^2 &= J_0^2(z) + \epsilon J_1^2(z) \end{aligned} \tag{4.40}$$

for operator (4.19).

Remark 4.1.5. Functions (4.40) are said to be functionally independent if $J^2 = \Psi(J^1)$, i.e., if the equation

$$J_0^2(z) + \epsilon J_1^2(z) = \Psi(J_0^1(z) + \epsilon J_1^1(z)) + o(\epsilon) \tag{4.41}$$

with a certain function Ψ holds identically in z. If such a function Ψ does not exist, the functions (4.40) are said to be functionally independent. It is manifest that if J_0^1 and J_0^2 are functionally independent, then so are the functions (4.40) as well.

4.2 Perturbed pseudo-parabolic PDE: Model IIIa

In this section we consider a perturbed pseudo-parabolic PDE (6)

$$u_t = ((u^\alpha + \epsilon u^\beta)(u + u_t)_x)_x, \tag{4.42}$$

where ϵ is a small parameter, while α and β are arbitrary constants. Firstly, we perform approximate symmetry analysis of a general case $\alpha \neq \beta$ for $\alpha, \beta > 0$. Secondly, we consider particular cases of (4.42) for different values of α and β which arises from the analysis of the general case. In each case the approximate symmetries are obtained and then used to perform symmetry reductions and/or construct group approximate invariant solutions.

4.2.1 Approximate symmetries

The generator of approximate symmetries of (4.42) is

$$\begin{aligned} X &= X_0 + \epsilon X_1 \\ &= \left(\xi^0 \frac{\partial}{\partial t} + \tau^0 \frac{\partial}{\partial x} + \eta^0 \frac{\partial}{\partial u}\right) + \epsilon \left(\xi^1 \frac{\partial}{\partial t} + \tau^1 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u}\right) \end{aligned} \tag{4.43}$$

if and only if

$$X^{[3]}(u_t - ((u^\alpha + \epsilon u^\beta)(u + u_t)_x)_x)\big|_{(4.42)} = o(\epsilon), \tag{4.44}$$

or equivalently

$$\begin{aligned}(X_0^{[3]}(u_t - (u^\alpha(u_x + u_{tx}))_x) + \epsilon(X_1^{[3]}(u_t - (u^\alpha(u_x + u_{tx}))_x)\\ + X_0^{[3]}((-u^\beta(u + u_t)_x)_x)))\big|_{(4.42)} = 0.\end{aligned} \tag{4.45}$$

The coefficients ξ^i, τ^i, and η^i $(i = 0, 1)$ are unknown functions of t, x, u and $X^{[3]}$ is the third prolongation of X. Equation (4.44) is the determining equation for infinitesimal approximate symmetries.

First step. **Calculation of symmetries, X_0, of the unperturbed equation**:
The symmetry operators X_0 of the unperturbed equation ($\epsilon = 0$)

$$u_t - (u^\alpha(u_x + u_{tx}))_x = 0 \tag{4.46}$$

are obtained by solving the determining equation for exact symmetries

$$X_0^{[3]}(u_t - (u^\alpha(u_x + u_{tx}))_x)\big|_{(4.46)} = 0, \tag{4.47}$$

where $X_0^{[3]}$ is the third prolongation of the vector field X_0 given by

$$X_0^{[3]} = X_0 + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}}. \tag{4.48}$$

From equation (4.47) we have

$$\begin{aligned}\big(\zeta_t - u^\alpha\zeta_{txx} - u^\alpha\zeta_{xx} - \alpha u^{\alpha-1}u_x\zeta_{tx} - \left(\alpha u^{\alpha-1}u_x + \alpha u^{\alpha-1}(u_x + u_{tx})\right)\zeta_x\\ - \left(\alpha(\alpha-1)u^{\alpha-2}u_x(u_x + u_{tx}) + \alpha u^{\alpha-1}(u_{xx} + u_{txx})\right)\eta\big)\big|_{(4.46)} = 0,\end{aligned} \tag{4.49}$$

where the coefficients ζ_t, ζ_x, ζ_{tx}, ζ_{xx} and ζ_{txx} are given respectively by equations (1.23), (1.24), (1.26), (1.27) and (2.6). When expanded , equation (4.49) yields an overdetermined system of linear homogeneous partial differential equations (*determining equations*) that can be solved for the coefficients ξ^0, τ^0 and η^0 of the approximate symmetry operator (4.43) using the classical Lie symmetry method.

With the aid of *YaLie* [7] software package, the determining equations become

$$\xi^0_u = 0, \tag{4.50}$$

$$\tau^0_u = 0, \tag{4.51}$$

$$\eta^0_{uu} = 0, \tag{4.52}$$

$$\xi^0_x = 0, \tag{4.53}$$

$$\eta^0_{xu} = 0, \tag{4.54}$$

$$\tau^0_t = 0, \tag{4.55}$$

$$\alpha u^{\alpha-1}\eta^0_x - u^\alpha \tau^0_{xx} = 0, \tag{4.56}$$

$$u^\alpha \eta^0_{xx} - \eta^0_t + u^\alpha \eta^0_{txx} = 0, \tag{4.57}$$

$$\alpha u^{\alpha-1}\eta^0 - 2u^\alpha \tau^0_x = 0, \tag{4.58}$$

$$\alpha u^{\alpha-1}\eta^0 - 2u^\alpha \tau^0_x + u^\alpha \xi^0_t + u^\alpha \eta^0_{tu} = 0, \tag{4.59}$$

$$(1-\alpha)\alpha u^{\alpha-2}\eta^0 - \alpha u^{\alpha-1}\eta^0_u + 2\alpha u^{\alpha-1}\tau^0_x = 0, \tag{4.60}$$

$$(1-\alpha)\alpha u^{\alpha-2}\eta^0 - \alpha u^{\alpha-1}\eta^0_u + 2\alpha u^{\alpha-1}\tau^0_x - \alpha u^{\alpha-1}\xi^0_t - \alpha u^{\alpha-1}\eta^0_{tu} = 0, \tag{4.61}$$

$$2\alpha u^{\alpha-1}\eta^0_x - u^\alpha \tau^0_{xx} + \alpha u^{\alpha-1}\eta^0_{tx} = 0, \tag{4.62}$$

where the subscripts denote partial derivatives with respect to the indicated variable. From equations (4.50) and (4.53) we have

$$\xi^0 = A(t), \tag{4.63}$$

where $A(t)$ is an arbitrary function of t. Equations (4.51) and (4.55) give

$$\tau^0 = B(x), \tag{4.64}$$

where $B(x)$ is an arbitrary function of x. Substituting equation (4.64) into equation (4.58) we obtain

$$\eta^0 = \frac{2uB'(x)}{\alpha}. \tag{4.65}$$

Equation (4.52) is identically satisfied by (4.65). Substituting equations (4.63), (4.64), and (4.65) into equations (4.54), (4.56), (4.57), (4.59), (4.60), (4.61) and (4.62) reduce to the following equations

$$B''(x) = 0, \tag{4.66}$$

$$A'(t) = 0. \tag{4.67}$$

From equations (4.66) and (4.67), we respectively obtain

$$B(x) = k_1 x + k_2, \tag{4.68}$$

$$A(t) = k_3, \tag{4.69}$$

where k_1, k_2 and k_3 are constants of integration. Thus,

$$X_0 = k_3\frac{\partial}{\partial t} + (k_1 x + k_2)\frac{\partial}{\partial x} + \frac{2k_1 u}{\alpha}\frac{\partial}{\partial u}. \tag{4.70}$$

Therefore the unperturbed equation (4.46) admits the three-dimensional Lie algebra with the basis

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}. \end{aligned} \tag{4.71}$$

Second step: We determine the auxiliary function H given by

$$H = \frac{1}{\epsilon}X_0^{[3]}\left(u_t - ((u^\alpha + \epsilon u^\beta)(u+u_t)_x)_x\right)\big|_{(4.42)}, \tag{4.72}$$

where $X_0^{[3]}$ is the third prolongation of X_0 given by

$$\begin{aligned} X_0^{[3]} = k_3\frac{\partial}{\partial t} + (k_1 x + k_2)\frac{\partial}{\partial x} + \frac{2k_1 u}{\alpha}\frac{\partial}{\partial u} + \zeta_t\frac{\partial}{\partial u_t} + \zeta_x\frac{\partial}{\partial u_x} + \zeta_{tx}\frac{\partial}{\partial u_{tx}} \\ +\zeta_{xx}\frac{\partial}{\partial u_{xx}} + \zeta_{txx}\frac{\partial}{\partial u_{txx}} \end{aligned} \tag{4.73}$$

and the coefficients ζ's are given by

$$\zeta_t = \frac{2k_1}{\alpha}u_t, \tag{4.74}$$

$$\zeta_x = \left(\frac{2k_1}{\alpha} - k_1\right)u_x, \tag{4.75}$$

$$\zeta_{tx} = \left(\frac{2k_1}{\alpha} - k_1\right)u_{tx}, \tag{4.76}$$

$$\zeta_{xx} = 2\left(\frac{k_1}{\alpha} - k_1\right)u_{xx}, \tag{4.77}$$

$$\zeta_{txx} = 2\left(\frac{k_1}{\alpha} - k_1\right)u_{txx}. \tag{4.78}$$

Substituting the operator (4.73) into (4.72) and simplifying we obtain the auxiliary function

$$H = \frac{2k_1(\alpha-\beta)u^{\beta-1}}{\alpha}(\beta u_x^2 + \beta u_x u_{tx} + u u_{xx} + u u_{txx}). \tag{4.79}$$

Third step: Now we calculate the operators X_1 by solving the inhomogeneous determining equation for deformations:

$$X_1^{[3]}(u_t-(u^\alpha(u_x+u_{tx}))_x)\big|_{(4.46)}+\frac{2k_1(\alpha-\beta)u^{\beta-1}}{\alpha}(\beta u_x^2+\beta u_x u_{tx}+uu_{xx}+uu_{txx})=0, \quad (4.80)$$

where $X_1^{[3]}$ is the third prolongation of X_1. Expansion of (4.80) yields an overdetermined system of linear inhomogeneous partial differential equations that can be solved for the coefficients ξ^1, τ^1, and η^1. The expansion of

$$X_1^{[3]}(u_t-(u^\alpha(u_x+u_{tx}))_x)\big|_{(4.46)} \quad (4.81)$$

is the same as that of equation (4.49) with superscript zero replaced by one. Thus, the determining equations of (4.80) become

$$\xi^1_u=0, \quad (4.82)$$

$$\tau^1_u=0, \quad (4.83)$$

$$\eta^1_{uu}=0, \quad (4.84)$$

$$\xi^1_x=0, \quad (4.85)$$

$$\eta^1_{xu}=0, \quad (4.86)$$

$$\tau^1_t=0, \quad (4.87)$$

$$\alpha u^{\alpha-1}\eta^1_x-u^\alpha\tau^1_{xx}=0, \quad (4.88)$$

$$u^\alpha\eta^1_{xx}-\eta^1_t+u^\alpha\eta^1_{txx}=0, \quad (4.89)$$

$$\alpha u^{\alpha-1}\eta^1-2u^\alpha\tau^1_x-\frac{2k_1(\alpha-\beta)u^\beta}{\alpha}=0, \quad (4.90)$$

$$\alpha u^{\alpha-1}\eta^1-2u^\alpha\tau^1_x+u^\alpha\xi^1_t+u^\alpha\eta^1_{tu}-\frac{2k_1(\alpha-\beta)u^\beta}{\alpha}=0, \quad (4.91)$$

$$(1-\alpha)\alpha u^{\alpha-2}\eta^1-\alpha u^{\alpha-1}\eta^1_u+2\alpha u^{\alpha-1}\tau^1_x+\frac{2k_1\beta(\alpha-\beta)u^{\beta-1}}{\alpha}=0, \quad (4.92)$$

$$(1-\alpha)\alpha u^{\alpha-2}\eta^1-\alpha u^{\alpha-1}\eta^1_u+2\alpha u^{\alpha-1}\tau^1_x-\alpha u^{\alpha-1}\xi^1_t-\alpha u^{\alpha-1}\eta^1_{tu} +\frac{2k_1\beta(\alpha-\beta)u^{\beta-1}}{\alpha}=0, \quad (4.93)$$

$$2\alpha u^{\alpha-1}\eta^1_x-u^\alpha\tau^1_{xx}+\alpha u^{\alpha-1}\eta^1_{tx}=0, \quad (4.94)$$

where the subscripts denote partial derivatives with respect to the indicated variable. From equations (4.82) and (4.85) we have

$$\xi^1=C(t), \quad (4.95)$$

where $C(t)$ is an arbitrary function of t. Equations (4.83) and (4.87) give

$$\tau^1 = D(x), \tag{4.96}$$

where $D(x)$ is an arbitrary function of x. Substituting equation (4.96) into equation (4.90) we obtain

$$\eta^1 = \frac{2uD'(x)}{\alpha} + \frac{2k_1(\alpha-\beta)u^{\beta-\alpha+1}}{\alpha^2}. \tag{4.97}$$

Substituting equation (4.97) into equation (4.84) we have

$$\eta^1_{uu} = \frac{2k_1(\alpha-\beta)(\beta-\alpha+1)(\beta-\alpha)u^{\beta-\alpha-1}}{\alpha^2} = 0 \tag{4.98}$$

which implies that

$$k_1 = 0 \tag{4.99}$$

or

$$\beta = \alpha - 1 \tag{4.100}$$

since $\alpha \neq \beta$. We now consider the two cases separately including the case $\alpha = \beta$.

Subcase 4.2.1. $k_1 = 0$.

When $k_1 = 0$, we therefore have

$$\eta^1 = \frac{2uD'(x)}{\alpha}. \tag{4.101}$$

Substituting equations (4.95), (4.96), (4.99) and (4.101) into (4.88), (4.89), (4.91), (4.92), (4.93) and (4.94) we get

$$D''(x) = 0, \tag{4.102}$$

$$C'(t) = 0. \tag{4.103}$$

From equations (4.102) and (4.103) we obtain

$$D(x) = d_1x + d_2, \tag{4.104}$$

$$C(t) = d_3, \tag{4.105}$$

where d_1, d_2, and d_3 are constants of integration. Thus,

$$X_1 = d_3\frac{\partial}{\partial t} + (d_1x + d_2)\frac{\partial}{\partial x} + \frac{2d_1u}{\alpha}\frac{\partial}{\partial u}. \tag{4.106}$$

Then, the approximate symmetries of (4.42) are obtained from

$$X = (k_3 + \epsilon d_3)\frac{\partial}{\partial t} + [k_2 + \epsilon(d_1 x + d_2)]\frac{\partial}{\partial x} + \epsilon\frac{2d_1 u}{\alpha}\frac{\partial}{\partial u}, \tag{4.107}$$

and are given by

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t},\\
\mathbf{v}_2 &= \frac{\partial}{\partial x},\\
\mathbf{v}_3 &= \epsilon\left(x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}\right),\\
\mathbf{v}_4 &= \epsilon\mathbf{v}_1,\\
\mathbf{v}_5 &= \epsilon\mathbf{v}_2.
\end{aligned} \tag{4.108}$$

Subcase 4.2.2. $\beta = \alpha - 1$.

When $\beta = \alpha - 1$ we therefore have

$$\eta^1 = \frac{2uD'(x)}{\alpha} + \frac{2k_1}{\alpha^2}. \tag{4.109}$$

Substitution of equations (4.95), (4.96), (4.100) and (4.109) into (4.88), (4.89), (4.91), (4.92), (4.93) and (4.94) we get

$$D''(x) = 0, \tag{4.110}$$

$$C'(t) = 0. \tag{4.111}$$

From equations (4.110) and (4.111) we obtain

$$D(x) = c_1 x + c_2, \tag{4.112}$$

$$C(t) = c_3, \tag{4.113}$$

where c_1, c_2, and c_3 are constants of integration. Thus,

$$X_1 = c_3\frac{\partial}{\partial t} + (c_1 x + c_2)\frac{\partial}{\partial x} + \left(\frac{2c_1 u}{\alpha} + \frac{2k_1}{\alpha^2}\right)\frac{\partial}{\partial u}. \tag{4.114}$$

Then, the approximate symmetries of (4.42) are obtained from

$$X = (k_3 + \epsilon c_3)\frac{\partial}{\partial t} + [k_1 x + k_2 + \epsilon(c_1 x + c_2)]\frac{\partial}{\partial x} + \left[\frac{2k_1 u}{\alpha} + \epsilon\left(\frac{2c_1 u}{\alpha} + \frac{2k_1}{\alpha^2}\right)\right]\frac{\partial}{\partial u}, \tag{4.115}$$

and are given by

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{v}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{v}_3 &= \epsilon\left(x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}\right), \\
\mathbf{v}_4 &= \epsilon\mathbf{v}_1, \\
\mathbf{v}_5 &= \epsilon\mathbf{v}_2 \\
\mathbf{v}_6 &= x\frac{\partial}{\partial x} + \left(\frac{2u}{\alpha} + \frac{2\epsilon}{\alpha^2}\right)\frac{\partial}{\partial u}.
\end{aligned}
\tag{4.116}
$$

Subcase 4.2.3. $\alpha = \beta$.

When $\alpha = \beta$ equation (4.42) becomes

$$
u_t = ((u^\alpha + \epsilon u^\alpha)(u + u_t)_x)_x \tag{4.117}
$$

and the approximate symmetries of (4.42) are given by

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{v}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{v}_3 &= x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}, \\
\mathbf{v}_4 &= \epsilon\mathbf{v}_1, \\
\mathbf{v}_5 &= \epsilon\mathbf{v}_2, \\
\mathbf{v}_6 &= \epsilon\mathbf{v}_3.
\end{aligned}
\tag{4.118}
$$

Subsubcase 4.2.1. $\alpha = \beta = 1$.

When $\alpha = \beta = 1$ equation (4.42) becomes

$$
u_t = ((u + \epsilon u)(u + u_t)_x)_x \tag{4.119}
$$

and the approximate symmetries of (4.42) are given by

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t},\\
\mathbf{v}_2 &= \frac{\partial}{\partial x},\\
\mathbf{v}_3 &= x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u},\\
\mathbf{v}_4 &= \epsilon \mathbf{v}_1,\\
\mathbf{v}_5 &= \epsilon \mathbf{v}_2,\\
\mathbf{v}_6 &= \epsilon \mathbf{v}_3.
\end{aligned} \tag{4.120}$$

Subsubcase 4.2.2. $\alpha = 0$ and $\beta = 1$.

When $\alpha = 0$ and $\beta = 1$ equation (4.42) becomes

$$u_t = ((1 + \epsilon u)(u + u_t)_x)_x \tag{4.121}$$

and the approximate symmetries of (4.42) are given by

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t},\\
\mathbf{v}_2 &= \frac{\partial}{\partial x},\\
\mathbf{v}_3 &= -\frac{\partial}{\partial t} + u\frac{\partial}{\partial u},\\
\mathbf{v}_4 &= \epsilon \mathbf{v}_1,\\
\mathbf{v}_5 &= \epsilon \mathbf{v}_2,\\
\mathbf{v}_6 &= \epsilon \mathbf{v}_3.
\end{aligned} \tag{4.122}$$

4.2.2 Symmetry reductions and approximately invariant solutions

In this section, approximate invariant solutions will be derived from particular linear combinations of the approximate symmetries obtained in the previous section. We will consider the two subsubcases, 4.2.1 and 4.2.2.

Subsubcase 4.2.1

In this subsubcase we will consider the combinations $\mathbf{v}_2+\mathbf{v}_6$, $\mathbf{v}_3+\mathbf{v}_5$, $\mathbf{v}_3+\mathbf{v}_4$ and $\mathbf{v}_1+\mathbf{v}_6$.

(a) Invariance under $\mathbf{v}_2 + \mathbf{v}_6$. The approximate invariants for $\mathbf{v}_2 + \mathbf{v}_6$ are determined by the equation

$$\left(\frac{\partial}{\partial x} + \epsilon\left(x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\right)\right)(J_0 + \epsilon J_1) = o(\epsilon), \tag{4.123}$$

equivalently

$$\frac{\partial}{\partial x}(J_0) = 0, \tag{4.124}$$

and

$$\frac{\partial}{\partial x}(J_1) + \left(x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}\right)(J_0) = 0. \tag{4.125}$$

From equation (4.124) we have characteristic equations

$$\frac{dt}{0} = \frac{dx}{1} = \frac{du}{0} \tag{4.126}$$

which give the two functionally independent solutions

$$J_0^1 = t \tag{4.127}$$

and

$$J_0^2 = u. \tag{4.128}$$

Substituting (4.127) into (4.125) we get

$$\frac{\partial}{\partial x}(J_1^1) = 0. \tag{4.129}$$

The simplest solution is

$$J_1^1 = 0, \tag{4.130}$$

and hence the first invariant is

$$J_1 = t. \tag{4.131}$$

Substituting (4.128) into (4.125) we get

$$\frac{\partial}{\partial x}(J_1^2) + 2u = 0, \tag{4.132}$$

and the corresponding characteristic equation is

$$\frac{dx}{1} = -\frac{dJ_1^2}{2u} \tag{4.133}$$

which gives

$$J_1^2 = -2xu + k. \tag{4.134}$$

Setting $k = 0$ we have

$$J_1^2 = -2xu \tag{4.135}$$

and hence the second invariant is

$$J_2 = u - \epsilon 2xu. \tag{4.136}$$

Thus, the approximately invariant solution $J_2 = f(J_1)$ is

$$u - \epsilon 2xu = f(t) \tag{4.137}$$

which implies that

$$\begin{aligned} u(t,x) &= (1 - 2\epsilon x)^{-1} f(t) \\ &\approx (1 + 2\epsilon x) f(t) + o(\epsilon^2). \end{aligned} \tag{4.138}$$

Therefore the approximate invariant solution is

$$u(t,x) = (1 + 2\epsilon x) f(t). \tag{4.139}$$

Substituting (4.139) into (4.119) we obtain the following ODE

$$f' + 2\epsilon x f' = 0 \tag{4.140}$$

where "′" denotes differentiation with respect to $J_1 = t$, and it implies that

$$f' = 0. \tag{4.141}$$

The solution of equation (4.141) is

$$f(t) = K_1 \tag{4.142}$$

where K_1 is a constant of integration. Hence the approximate invariant solution is given by

$$u(t,x) = (1 + 2\epsilon x) K_1. \tag{4.143}$$

(b) Invariance under $\mathbf{v}_3 + \mathbf{v}_5$. The approximate invariants for $\mathbf{v}_3 + \mathbf{v}_5$ are determined by the equations

$$\left(x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u} \right)(J_0) = 0 \tag{4.144}$$

and

$$\left(x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}\right)(J_1)+\frac{\partial}{\partial x}(J_0)=0. \tag{4.145}$$

Equation (4.144) has two functionally independent solutions $J_0^1 = t$ and $J_0^2 = u/x^2$. The simplest solutions of (4.145) are $J_1^1 = 0$ and $J_1^2 = -2u/x^3$. Therefore we have two independent invariants $J_1 = t$ and $J_2 = u/x^2 - 2\epsilon u/x^3$. Thus, the approximate invariant solution $J_2 = g(J_1)$ is given by

$$\frac{u}{x^2}-\frac{2\epsilon u}{x^3}=g(t) \tag{4.146}$$

which implies that

$$\begin{aligned} u(t,x) &= x^2\left(1-\frac{2\epsilon}{x}\right)^{-1} g(t) \\ &\approx x^2\left(1+\frac{2\epsilon}{x}\right) g(t)+o(\epsilon^2). \end{aligned} \tag{4.147}$$

Hence the approximate invariant solution is

$$u(t,x)=x^2\left(1+\frac{2\epsilon}{x}\right) g(t). \tag{4.148}$$

Substituting (4.148) into (4.119) we obtain the following ODE

$$x^2g'-6x^2g(g+g')+\epsilon(2xg'-2(6x+3x^2)g(g+g'))=0 \tag{4.149}$$

where "′" denotes differentiation with respect to $J_1 = t$.

(c) Invariance under $\mathbf{v}_3+\mathbf{v}_4$. The approximate invariants for $\mathbf{v}_3+\mathbf{v}_4$ are determined by the equations

$$\left(x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}\right)(J_0)=0 \tag{4.150}$$

and

$$\left(x\frac{\partial}{\partial x}+2u\frac{\partial}{\partial u}\right)(J_1)+\frac{\partial}{\partial t}(J_0)=0. \tag{4.151}$$

Equation (4.150) has two functionally independent solutions $J_0^1 = t$ and $J_0^2 = u/x^2$. The simplest solutions of (4.151) are $J_1^1 = -\ln x$ and $J_1^2 = 0$. Therefore we have two independent invariants $J_1 = t - \epsilon \ln x$ and $J_2 = u/x^2$. Thus, the approximate invariant solution $J_2 = h(J_1)$ is given by

$$x^{-2}u = h(t-\epsilon\ln x). \tag{4.152}$$

Hence the approximate invariant solution is

$$u(t, x) = x^2 h(t - \epsilon \ln x). \tag{4.153}$$

Substituting (4.153) into (4.119) we obtain the following ODE

$$6h^2 - h' + 6hh' + \epsilon(6h^2 - hh' - 2h'^2 - 5hh'') = 0 \tag{4.154}$$

where "′" denotes differentiation with respect to $J_1 = t - \epsilon \ln x$.

Subsubcase 4.2.2

In this subsubcase we will consider the combinations $\mathbf{v}_2 + \mathbf{v}_6$ and $\mathbf{v}_1 + \mathbf{v}_6$.

(a) Invariance under $\mathbf{v}_2 + \mathbf{v}_6$. The approximate invariants for $\mathbf{v}_2 + \mathbf{v}_6$ are determined by the equations

$$\frac{\partial}{\partial x}(J_0) = 0, \tag{4.155}$$

and

$$\frac{\partial}{\partial x}(J_1) + \left(-\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}\right)(J_0) = 0. \tag{4.156}$$

Equation (4.155) has two functionally independent solutions $J_0^1 = t$ and $J_0^2 = u$. The simplest solutions of (4.156) are $J_1^1 = x$ and $J_1^2 = -xu$. Therefore we have two independent invariants $J_1 = t + \epsilon x$ and $J_2 = u - \epsilon xu$. Thus, the approximate invariant solution $J_2 = F(J_1)$ is given by

$$u - \epsilon xu = F(t + \epsilon x) \tag{4.157}$$

which implies that

$$\begin{aligned} u(t, x) &= (1 - \epsilon x)^{-1} F(t + \epsilon x) \\ &\approx (1 + \epsilon x) F(t + \epsilon x) + o(\epsilon^2). \end{aligned} \tag{4.158}$$

Hence the approximate invariant solution is

$$u(t, x) = (1 + \epsilon x) F(t + \epsilon x). \tag{4.159}$$

Substituting (4.159) into (4.119) we obtain the following ODE

$$F' + \epsilon x F' = 0 \tag{4.160}$$

where "$'$" denotes differentiation with respect to $J_1 = t - \epsilon x$, and it implies that

$$F' = 0. \tag{4.161}$$

The solution of equation (4.161) is

$$F = K_2 \tag{4.162}$$

where K_2 is a constant of integration. Hence the approximate invariant solution is given by

$$u(t, x) = (1 + \epsilon x)\, K_2. \tag{4.163}$$

(b) Invariance under $\mathbf{v}_1 + \mathbf{v}_6$. The approximate invariants for $\mathbf{v}_1 + \mathbf{v}_6$ are determined by the equations

$$\frac{\partial}{\partial t}(J_0) = 0 \tag{4.164}$$

and

$$\frac{\partial}{\partial t}(J_1) + \left(-\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}\right)(J_0) = 0. \tag{4.165}$$

Equation (4.164) has two functionally independent solutions $J_0^1 = x$ and $J_0^2 = u$. The simplest solutions of (4.165) are $J_1^1 = 0$ and $J_1^2 = -tu$. Therefore we have two independent invariants $J_1 = x$ and $J_2 = u - \epsilon t u$. Thus, the approximate invariant solution $J_2 = G(J_1)$ is given by

$$u - \epsilon t u = G(x) \tag{4.166}$$

which implies that

$$\begin{aligned} u(t, x) &= (1 - \epsilon t)^{-1}\, G(x) \\ &\approx (1 + \epsilon t)\, G(x) + o(\epsilon^2). \end{aligned} \tag{4.167}$$

Hence the approximate invariant solution is

$$u(t, x) = (1 + \epsilon t)\, G(x). \tag{4.168}$$

Substituting (4.168) into (4.119) we obtain the following ODE

$$G'' + \epsilon(G'^2 + G'' + tG'' + GG'' - G) = 0 \tag{4.169}$$

where "$'$" denotes differentiation with respect to $J_1 = x$.

4.3 Perturbed pseudo-parabolic PDE: Model IIIb

In this section, for completeness we consider a perturbed pseudo-parabolic PDE (7)

$$u_t = ((u^\alpha + \epsilon u^\beta)u_x)_x, \tag{4.170}$$

where ϵ is a small parameter, while α and β are arbitrary constants. Firstly, we perform approximate symmetry analysis of a general case $\alpha \neq \beta$ for $\alpha, \beta > 0$. Secondly, we consider particular cases of (4.170) for different values of α and β which arises from the analysis of the general case. In each case the approximate symmetries are obtained and then used to perform symmetry reductions and/or construct group approximate invariant solutions.

4.3.1 Approximate symmetries

The generator of approximate symmetries of (4.170) is

$$\begin{aligned} X &= X_0 + \epsilon X_1 \\ &= \left(\xi^0 \frac{\partial}{\partial t} + \tau^0 \frac{\partial}{\partial x} + \eta^0 \frac{\partial}{\partial u}\right) + \epsilon \left(\xi^1 \frac{\partial}{\partial t} + \tau^1 \frac{\partial}{\partial x} + \eta^1 \frac{\partial}{\partial u}\right) \end{aligned} \tag{4.171}$$

if and only if

$$X^{[2]}(u_t - ((u^\alpha + \epsilon u^\beta)u_x)_x)\big|_{(4.170)} = o(\epsilon), \tag{4.172}$$

or equivalently

$$\left(X_0^{[2]}(u_t - (u^\alpha u_x)_x) + \epsilon(X_1^{[2]}(u_t - (u^\alpha u_x)_x) + X_0^{[2]}(-(u^\beta u_x)_x))\right)\big|_{(4.170)} = 0. \tag{4.173}$$

The coefficients ξ^i, τ^i, and η^i $(i = 0, 1)$ are unknown functions of t, x, u and $X^{[2]}$ is the second prolongation of X. Equation (4.172) is the determining equation for infinitesimal approximate symmetries.

First step. Calculation of symmetries, X_0, of the unperturbed equation: The symmetry operators X_0 of the unperturbed equation ($\epsilon = 0$)

$$u_t - (u^\alpha u_x)_x = 0 \tag{4.174}$$

are obtained by solving the determining equation for exact symmetries

$$X_0^{[2]}(u_t - (u^\alpha u_x)_x)\big|_{(4.174)} = 0, \tag{4.175}$$

where $X_0^{[2]}$ is the second prolongation of the vector field X_0 given by

$$X_0^{[2]} = X_0 + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xx} \frac{\partial}{\partial u_{xx}}. \tag{4.176}$$

From equation (4.175) we have

$$\left(\zeta_t - u^\alpha \zeta_{xx} - 2\alpha u^{\alpha-1} u_x \zeta_x - \left(\alpha(\alpha-1)u^{\alpha-2}u_x^2 + \alpha u^{\alpha-1}u_{xx}\right)\eta\right)\Big|_{(4.174)} = 0, \tag{4.177}$$

where the coefficients ζ_t, ζ_x and ζ_{xx} are given respectively by equations (1.23), (1.24) and (1.27). When expanded , equation (4.177) yields an overdetermined system of linear homogeneous partial differential equations (*determining equations*) that can be solved for the coefficients ξ^0, τ^0 and η^0 of the approximate symmetry operator (4.171) using the classical Lie symmetry method.

With the aid of *YaLie* [7] software package, the determining equations become

$$\xi_u^0 = 0, \tag{4.178}$$

$$\tau_u^0 = 0, \tag{4.179}$$

$$\xi_x^0 = 0, \tag{4.180}$$

$$\eta_t^0 - u^\alpha \eta_{xx}^0 = 0, \tag{4.181}$$

$$\alpha(1-\alpha)u^{\alpha-2}\eta^0 - \alpha u^{\alpha-1}\eta_u^0 + 2\alpha u^{\alpha-1}\tau_x^0 - \alpha u^{\alpha-1}\xi_t^0 - u^\alpha \eta_{uu}^0 = 0, \tag{4.182}$$

$$-\alpha u^{\alpha-1}\eta^0 + 2u^\alpha \tau_x^0 - u^\alpha \xi_t^0 = 0, \tag{4.183}$$

$$-2\alpha u^{\alpha-1}\eta_x^0 + u^\alpha \tau_{xx}^0 - 2u^\alpha \eta_{xu}^0 - \tau_t^0 = 0, \tag{4.184}$$

where the subscripts denote partial derivatives with respect to the indicated variable. From equations (4.178) and (4.180) we have

$$\xi^0 = A(t), \tag{4.185}$$

where $A(t)$ is an arbitrary function of t. From equation (4.179) we get

$$\tau^0 = B(t,x), \tag{4.186}$$

where $B(t,x)$ is an arbitrary function of t and x. Substituting equations (4.185) and (4.186) into equation (4.183) we obtain

$$\eta^0 = \frac{u(2B_x - A_t)}{\alpha}. \tag{4.187}$$

Equation (4.182) is identically satisfied by (4.185), (4.186) and (4.187). Substituting equation (4.187) into equations (4.181) and (4.184) reduce to the following equations

$$A_{tt} + 2u^{\alpha} B_{xxx} - 2B_{tx} = 0, \tag{4.188}$$

$$(4 + 3\alpha) u^{\alpha} B_{xx} + \alpha B_t = 0. \tag{4.189}$$

Separating (4.189) by powers of u we have

$$B_t = 0 \text{ or } \alpha = 0 \tag{4.190}$$

and

$$B_{xx} = 0 \text{ or } 4 + 3\alpha = 0. \tag{4.191}$$

For $\alpha \neq 0, -4/3$ we get

$$B(x) = k_1 x + k_2, \tag{4.192}$$

where k_1 and k_2 are constants of integration. Substituting (4.192) into (4.188) we have

$$A_{tt} = 0 \tag{4.193}$$

which solves to

$$A(t) = k_3 t + k_4, \tag{4.194}$$

where k_3 and k_4 are constants of integration. Thus,

$$X_0 = (k_3 t + k_4)\frac{\partial}{\partial t} + (k_1 x + k_2)\frac{\partial}{\partial x} + \frac{(2k_1 - k_3)u}{\alpha}\frac{\partial}{\partial u}. \tag{4.195}$$

Therefore the unperturbed equation (4.174) admits the four-dimensional Lie algebra with the basis

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}, \\ X_4 &= t\frac{\partial}{\partial t} - \frac{u}{\alpha}\frac{\partial}{\partial u}. \end{aligned} \tag{4.196}$$

Second step: We determine the auxiliary function H given by

$$H = \frac{1}{\epsilon} X_0^{[2]} \left(u_t - ((u^{\alpha} + \epsilon u^{\beta}) u_x)_x \right) \big|_{(4.170)}, \tag{4.197}$$

where $X_0^{[2]}$ is the second prolongation of X_0 given by

$$X_0^{[2]} = (k_3 t + k_4)\frac{\partial}{\partial t} + (k_1 x + k_2)\frac{\partial}{\partial x} + \frac{(2k_1 - k_3)u}{\alpha}\frac{\partial}{\partial u} + \zeta_t \frac{\partial}{\partial u_t} + \zeta_x \frac{\partial}{\partial u_x} + \zeta_{xx}\frac{\partial}{\partial u_{xx}} \qquad (4.198)$$

and the coefficients ζ's are given by

$$\zeta_t = \left(\frac{2k_1 - k_3}{\alpha} - k_3\right) u_t, \qquad (4.199)$$

$$\zeta_x = \left(\frac{2k_1 - k_3}{\alpha} - k_3\right) u_x, \qquad (4.200)$$

$$\zeta_{xx} = \left(\frac{2k_1 - k_3}{\alpha} - 2k_1\right) u_{xx}. \qquad (4.201)$$

Substituting the operator (4.198) into (4.197) and simplifying we obtain the auxiliary function

$$H = \frac{(2k_1 - k_3)(\alpha - \beta)u^{\beta-1}}{\alpha}(\beta u_x^2 + u u_{xx}). \qquad (4.202)$$

Third step: Now we calculate the operators X_1 by solving the inhomogeneous determining equation for deformations:

$$X_1^{[2]}(u_t - (u^\alpha u_x)_x)\big|_{(4.174)} + \frac{(2k_1 - k_3)(\alpha - \beta)u^{\beta-1}}{\alpha}(\beta u_x^2 + u u_{xx}) = 0, \qquad (4.203)$$

where $X_1^{[2]}$ is the second prolongation of X_1. Expansion of (4.203) yields an overdetermined system of linear inhomogeneous partial differential equations that can be solved for the coefficients ξ^1, τ^1, and η^1. The expansion of expression

$$X_1^{[2]}(u_t - (u^\alpha u_x)_x)\big|_{(4.174)}$$

is the same as that in equation (4.177) with superscript zero replaced by one. Thus, the determining equations of (4.203) become

$$\xi_u^1 = 0, \qquad (4.204)$$

$$\tau_u^1 = 0, \qquad (4.205)$$

$$\xi_x^1 = 0, \qquad (4.206)$$

$$\eta_t^1 - u^\alpha \eta_{xx}^1 = 0, \qquad (4.207)$$

$$\alpha(1-\alpha)u^{\alpha-2}\eta^1 - \alpha u^{\alpha-1}\eta_u^1 + 2\alpha u^{\alpha-1}\tau_x^1 - \alpha u^{\alpha-1}\xi_t^1 - u^\alpha \eta_{uu}^1 + \frac{(2k_1 - k_3)(\alpha - \beta)\beta u^{\beta-1}}{\alpha} = 0, \qquad (4.208)$$

$$-\alpha u^{\alpha-1}\eta^1 + 2u^\alpha \tau^1_x - u^\alpha \xi^1_t + \frac{(2k_1-k_3)(\alpha-\beta)u^\beta}{\alpha} = 0, \tag{4.209}$$

$$-2\alpha u^{\alpha-1}\eta^1_x + u^\alpha \tau^1_{xx} - 2u^\alpha \eta^1_{xu} - \tau^1_t = 0, \tag{4.210}$$

where the subscripts denote partial derivatives with respect to the indicated variable. From equations (4.204) and (4.206) we have

$$\xi^1 = C(t), \tag{4.211}$$

where $C(t)$ is an arbitrary function of t. From equation (4.210) we get

$$\tau^1 = D(t,x), \tag{4.212}$$

where $D(t,x)$ is an arbitrary function of t and x. Substituting equations (4.211) and (4.212) into equation (4.209) we obtain

$$\eta^1 = \frac{u(2D_x - C_t)}{\alpha} + \frac{(2k_1-k_3)(\alpha-\beta)u^{\beta-\alpha+1}}{\alpha^2}. \tag{4.213}$$

Substituting equation (4.213) into equations (4.207) and (4.210) reduce to the following equations

$$C_{tt} + 2u^\alpha D_{xxx} - 2D_{tx} = 0, \tag{4.214}$$

$$(4+3\alpha)u^\alpha D_{xx} + \alpha D_t = 0. \tag{4.215}$$

Separating (4.215) by powers of u we have

$$D_t = 0 \text{ or } \alpha = 0 \tag{4.216}$$

and

$$D_{xx} = 0 \text{ or } 4+3\alpha = 0. \tag{4.217}$$

For $\alpha \neq 0, -4/3$ we get

$$D(x) = d_1 x + d_2, \tag{4.218}$$

where d_1 and d_2 are constants of integration. Substituting (4.218) into (4.214) we have

$$C_{tt} = 0 \tag{4.219}$$

which solves to

$$C(t) = d_3 t + d_4, \tag{4.220}$$

where d_3 and d_4 are constants of integration. Substituting (4.213), (4.218) and (4.220) into (4.208) we have

$$\frac{(2k_1 - k_3)(\alpha - \beta)(\beta - \alpha + 1)(\beta - \alpha)u^{\beta-\alpha-1}}{\alpha^2} = 0 \tag{4.221}$$

which implies that

$$2k_1 = k_3 \tag{4.222}$$

or

$$\beta = \alpha - 1 \tag{4.223}$$

since $\alpha \neq \beta$. We now consider the two cases separately including the case $\alpha = \beta$.

Subcase 4.3.1. $2k_1 = k_3$.

When $2k_1 = k_3$, we therefore have

$$\eta^1 = \frac{(2d_1 - d_3)u}{\alpha}. \tag{4.224}$$

Thus,

$$X_1 = (d_3 t + d_4)\frac{\partial}{\partial t} + (d_1 x + d_2)\frac{\partial}{\partial x} + \frac{(2d_1 - d_3)u}{\alpha}\frac{\partial}{\partial u}. \tag{4.225}$$

Then, the approximate symmetries of (4.170) are obtained from

$$\begin{aligned} X = [2k_1 t + k_4 + \epsilon(d_3 t + d_4)]\frac{\partial}{\partial t} + [k_1 x + k_2 + \epsilon(d_1 x + d_2)]\frac{\partial}{\partial x} \\ + \epsilon\frac{(2d_1 - d_3)u}{\alpha}\frac{\partial}{\partial u}, \end{aligned} \tag{4.226}$$

and are given by

$$\begin{aligned} \mathbf{v}_1 &= \frac{\partial}{\partial t}, \\ \mathbf{v}_2 &= \frac{\partial}{\partial x}, \\ \mathbf{v}_3 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ \mathbf{v}_4 &= \epsilon\mathbf{v}_1, \\ \mathbf{v}_5 &= \epsilon\mathbf{v}_2, \\ \mathbf{v}_6 &= \epsilon\left(t\frac{\partial}{\partial t} - \frac{u}{\alpha}\frac{\partial}{\partial u}\right), \\ \mathbf{v}_7 &= \epsilon\left(x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}\right). \end{aligned} \tag{4.227}$$

Subcase 4.3.2. $\beta = \alpha - 1$.

When $\beta = \alpha - 1$ we therefore have

$$\eta^1 = \frac{(2d_1 - d_3)u}{\alpha} + \frac{2k_1 - k_3}{\alpha^2}. \tag{4.228}$$

Thus,

$$X_1 = (d_3 t + d_4)\frac{\partial}{\partial t} + (d_1 x + d_2)\frac{\partial}{\partial x} + \left(\frac{(2d_1 - d_3)u}{\alpha} + \frac{2k_1 - k_3}{\alpha^2}\right)\frac{\partial}{\partial u}. \tag{4.229}$$

Then, the approximate symmetries of (4.170) are obtained from

$$\begin{aligned} X = {} & [k_3 t + k_4 + \epsilon(d_3 t + d_4)]\frac{\partial}{\partial t} + [k_1 x + k_2 + \epsilon(d_1 x + d_2)]\frac{\partial}{\partial x} \\ & + \left[\frac{(2k_1 - k_3)u}{\alpha} + \epsilon\left(\frac{(2d_1 - d_3)u}{\alpha} + \frac{2k_1 - k_3}{\alpha^2}\right)\right]\frac{\partial}{\partial u}, \end{aligned} \tag{4.230}$$

and are given by

$$\begin{aligned} \mathbf{v}_1 &= \frac{\partial}{\partial t}, \\ \mathbf{v}_2 &= \frac{\partial}{\partial x}, \\ \mathbf{v}_3 &= t\frac{\partial}{\partial t} - \left(\frac{u}{\alpha} + \frac{\epsilon}{\alpha^2}\frac{\partial}{\partial u}\right), \\ \mathbf{v}_4 &= x\frac{\partial}{\partial x} + \left(\frac{2u}{\alpha} + \frac{2\epsilon}{\alpha^2}\frac{\partial}{\partial u}\right), \\ \mathbf{v}_5 &= \epsilon\mathbf{v}_1, \\ \mathbf{v}_6 &= \epsilon\mathbf{v}_2, \\ \mathbf{v}_7 &= \epsilon\left(t\frac{\partial}{\partial t} - \frac{u}{\alpha}\frac{\partial}{\partial u}\right), \\ \mathbf{v}_8 &= \epsilon\left(x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}\right). \end{aligned} \tag{4.231}$$

Subcase 4.3.3. $\alpha = \beta$.

When $\alpha = \beta$ equation (4.170) becomes

$$u_t = ((u^\alpha + \epsilon u^\alpha)u_x)_x \tag{4.232}$$

and the approximate symmetries of (4.170) are given by

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{v}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{v}_3 &= t\frac{\partial}{\partial t} - \frac{u}{\alpha}\frac{\partial}{\partial u}, \\
\mathbf{v}_4 &= x\frac{\partial}{\partial x} + \frac{2u}{\alpha}\frac{\partial}{\partial u}, \\
\mathbf{v}_5 &= \epsilon \mathbf{v}_1, \\
\mathbf{v}_6 &= \epsilon \mathbf{v}_2, \\
\mathbf{v}_7 &= \epsilon \mathbf{v}_3, \\
\mathbf{v}_8 &= \epsilon \mathbf{v}_4.
\end{aligned}
\tag{4.233}
$$

Subcase 4.3.4. $\alpha = -4/3$.

When $\alpha = -4/3$ equation (4.170) becomes

$$u_t = ((u^{-4/3} + \epsilon u^{\beta})u_x)_x \tag{4.234}$$

and the approximate symmetries of (4.170) are given by

$$
\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial t}, \\
\mathbf{v}_2 &= \frac{\partial}{\partial x}, \\
\mathbf{v}_3 &= t\frac{\partial}{\partial t} + \frac{3u}{4}\frac{\partial}{\partial u}, \\
\mathbf{v}_4 &= x\frac{\partial}{\partial x} - \frac{3u}{2}\frac{\partial}{\partial u}, \\
\mathbf{v}_5 &= x^2\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \\
\mathbf{v}_6 &= \epsilon \mathbf{v}_1, \\
\mathbf{v}_7 &= \epsilon \mathbf{v}_2, \\
\mathbf{v}_8 &= \epsilon \mathbf{v}_3, \\
\mathbf{v}_9 &= \epsilon \mathbf{v}_4, \\
\mathbf{v}_{10} &= \epsilon \mathbf{v}_5.
\end{aligned}
\tag{4.235}
$$

4.4 Conclusion

In this chapter, approximate symmetry analysis of perturbed PDEs was presented. Approximate symmetries of each submodel were obtained and particular linear combinations of approximate symmetries used to construct approximate invariant solutions.

CONCLUSION

In this work, both Lie group analysis and approximate symmetry analysis were employed to study different submodels of a pseudo-parabolic PDE modelling solvent uptake in polymeric solids. In chapter two, Lie point symmetries of of the pseudo-parabolic PDE for power law in diffusion coefficient with constant velocity were obtained. Corresponding optimal systems of one-dimensional subalgebras were derived and used to perform symmetry reductions and construct invariant solutions. In chapter three, Lie point symmetries of the pseudo-parabolic PDE for law in diffusion coefficient and viscosity were obtained. Optimal systems of one-dimensional subalgebras were derived and used to perform symmetry reductions and construct group invariant solutions.

In chapter four, approximate symmetries of the perturbed pseudo-parabolic PDE with diffusion coefficient with a perturbation parameter and constant viscosity were obtained. Some linear combinations of the approximate symmetries were used to construct approximate invariant solutions.

Biblography

[1] T. Alfrey, F. Gurnee and W. Lloyd. Diffusion in glassy polymers. *Journal of Polymer Science: Part C*, 12:249–261, 1966.

[2] V. A. Baikov, R. K. Gazizov and N. H. Ibragimov. Approximate symmetries. *Matematicacheskii Sbornik*, 64(2):427, 1989.

[3] V. A. Baikov, R. K. Gazizov and N. H. Ibragimov. Perturbation methods in group analysis. *Journal of Soviet Mathematics*, 55(1):1450–1485, 1991.

[4] G. W. Bluman and S. C. Anco. *Symmetry and Integration Methods for Differential Equations*. Springer-Verlag, New York, 2002.

[5] G. W. Bluman and S. Kumei. *Symmetries and Differential Equations*. Springer-Verlag New York Inc, 1989.

[6] B. J. Cantwell. *Introduction to Symmetry Analysis*. Cambridge University Press, Cambridge, 2002.

[7] J. M. Diaz. Short Guide to Yalie. Yet another Mathematica package for Lie symmetries. http://library.wolfram/infocenter/MathSource/4231/Yalie.ps.

[8] W. P. Düll. *Theory of a pseudoparabolic partial differential equation modeling solvent uptake in polymeric solids*. PhD thesis, Rheinischen Friedrich-Wilhelms-Universitat, 2004.

[9] C. J. Durning, D. A. Edwards and D. S. Cohen. Perturbation analysis of Thomas and Windle's model of case II transport. *AIChE Journal*, 42:2025–2035, 1996.

[10] D. A. Edwards and D. S. Cohen. A mathematical model for a dissolving polymer. *AIChE Journal*, 41:2345–2355, 1995.

[11] W. I. Fushchich and V. M. Shtelen. On approximate symmetry and approximate solutions of the non-linear wave equation with a small parameter. *Journal of Physics A: Mathematical and General*, 22:887–890, 1989.

[12] C. Y. Hui, K. C. Wu, R. C. Lasky and E. J. Kramer. Case-II diffusion in polymers. II. steady-state front motion. *Journal of Applied Physics*, 61:2345–2355, 1987.

[13] J. Hulshof and J. R. King. Analysis of a darcy flow model with a dynamic pressure saturation relation. *SIAM Journal on Applied Mathematics*, 59:318–346, 1998.

[14] P. E. Hydon. *Symmetry Methods for Differential Equations, A Beginner's Guide*. Cambridge University Press, New York, 2010.

[15] N. H. Ibragimov. *Elementary Lie Group Analysis and Ordinary Differential Equations*. John Wiley & Sons, Chichester, 1999.

[16] N. H. Ibragimov and V. F. Kovalev. *Approximate and renormgroup symmetries*. Beijing (P.R.China): Higher Education Press, 2009.

[17] N. H. Ibragimov (Ed.). *CRC Handbook of Lie Group Analysis of Differential Equations, Volume 1-3*. CRC Press, Boca Raton, Florida, 1994-1996.

[18] M. Molati. *Group Classification of Coupled Partial Differential Equations with Aplication to Flow in a Colapsible Channel and Diffusion Processes*. PhD thesis, University of the Witwatersrand, 2010.

[19] F. Oliveri. Lie symmetries of differential equations: Classical results and recent contributions. *Symmetry*, 2:658–706, 2010.

[20] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag New York Inc, 1986.

[21] L. V. Ovsiannikov. *Group Analysis of Differential Equations*. Academic Press, New York, 1982.

[22] J. Patera and P. Winternitz. Subalgebras of real three- and four-dimensional lie algebras. *Journal of Mathematical Physics*, 18:1449, 1977.

[23] H. Stephani. *Symmetry and Integration Methods for Differential Equations.* Springer-Verlag, New York, 2002.

[24] N. L. Thomas and A. H. Windle. A theory of case II diffusion. *Polymer*, 23:529–542, 1982.

[25] Wolfram Research Inc. *Mathematica, Version 7.0.* Wolfram Research Inc.: Champaign, Illinois, 2008.