Application of Lie Symmetry Method in Pricing Arithmetic Asian Options

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Declaration

I, Monts'uoe Edward Kubeletsane, student number 201902922, declare that this project entitled, *Application of Lie Symmetry Method in Pricing Arithmetic Asian Options* submitted for the degree of Bachelor of Science Honours in Applied Mathematics at National University of Lesotho has not been previously submitted by me at this or any other University. Further, I declare that this is my original work and any work done by others has been acknowledged in accordance.

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Abstract

This research investigates the application of Lie symmetry method to find analytic solutions for arithmetic Asian options, which are crucial financial derivatives for managing risk in various commodity markets. By employing a two state partial differential equation approach, the study uses Lie symmetry method to enhance option pricing models. The research involves finding determining equations, infinitesimal generators, and invariant solutions, as well as examining the influence of parameters such as volatility, interest rates, and time on option prices.

Keywords

Arithmetic Asian options
Risk management
Commodity markets
Lie symmetry
Option pricing models
Financial derivatives
Interest rates
Invariant solutions
Partial differential equation

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Dedication

To my family and friends.

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List of Abbreviations

PDE, Partial Differential Equation;
ODE, Ordinary Differential Equation;
DE, Differential Equation;
w.r.t, with respect to.

Chapter 1

Literature review

1.1 Introduction

Arithmetic Asian options, also known as *Average options*, are derivatives whose valuation depends on the average price of an underlying asset over a defined period (See [1, 2]). "Arithmetic" describes the process that is employed to determine the average price. It specifically entails adding up all of the underlying asset's prices throughout the specified period and dividing the result by the total number of prices. As a consequence, an average price is calculated with equal weights for every observation[3].

Arithmetic Asian options play a vital role in managing risk in commodity markets, particularly those influenced by commodities like crude oil, agricultural products [4], and precious metals. Multinational corporations, such as those in the food industry relying on commodities like wheat, can utilize these options to hedge against price fluctuations, ensuring stable production costs[5, 6].

Moreover, Arithmetic Asian options have broader implications, impacting consumer prices and household budgets [7]. For instance, fluctuations in commodity prices, such as crude oil [8, 9], directly affect everyday expenses like transportation costs. Thus, airlines can employ these options to mitigate fuel price risks and maintain stable ticket prices. Additionally, in the realm of precious metals like gold, these options enable investors to capitalize on long-term price trends while mitigating short-term volatility risks[10]. Jewelry manufacturers also leverage Arithmetic Asian options to manage raw material costs, ensuring consistent pricing for consumers despite fluctuating gold prices.

In summary, Arithmetic Asian options serve as indispensable risk management tools across various commodity markets, providing stability, predictability, and financial security for businesses and consumers alike, thus making their pricing important.

1.2 Background to the study

1.2.1 Lie Symmetry

Lie symmetry is a mathematical technique used to find exact solutions for DEs by finding transformations that keep the equations unchanged. It originates from the work of Marius Sophus Lie [11, 12, 13] and Élie-Joseph Cartan [14, 15] in the late 19th and early 20th centuries. It has found extensive applications in various fields, including physics [16, 17, 18], engineering [19, 20], and finance [21, 22, 23, 24, 25, 26].

In the context of option pricing, Lie symmetry analysis has been effectively applied, as demonstrated by studies such as Yue and Shen [27] on the fractal bond-pricing model, and Ramoeletsi's work [28] on the fractional Black-Scholes option pricing model. By employing Lie symmetry methods, researchers find complete Lie symmetry groups, infinitesimal generators, and exact invariant solutions for pricing models, indicating the versatility of Lie symmetry method in solving complex financial mathematical problems and potentially improving pricing accuracy in financial markets.

Moreover, Lie symmetry analysis has been extensively applied in the realm of mathematical finance, as evidenced by Kaibe and O'Hara[29] and Kaibe's PhD thesis[30] from the University of Essex. They demonstrated its utility in deriving exact invariant solutions for zero-coupon bond pricing equations, expanding the arsenal of pricing models for interest rate derivatives. Kaibe's thesis further delves into the application of Lie symmetry analysis to solve PDE associated with interest rate derivatives, providing benchmarks for testing numerical methods in financial markets and addressing issues such as negative interest rates in traditional models like the Vasicek model.

1.2.2 Financial Markets

Pricing and options are fundamental aspects of financial markets, influencing investment decisions, risk management, and market efficiency [31, 32].

Pricing

This is the process of determining the value of a financial instrument. Traditional pricing models, such as the Capital Asset Pricing Model (CAPM)[33, 34]; developed by Sharpe [35] and Lintner [36] ,and the Arbitrage Pricing Theory (APT) [37, 38], provides frameworks for valuing assets based on their risk and return characteristics. However, in recent years, there has been growing interest in alternative pricing models that incorporate additional factors and market features [39, 40, 41]. Factor-based models, such as the Fama-French five-factor model [42] and the Carhart four-factor model [43], consider additional factors beyond market risk, such as size, value, and momentum, to better explain asset returns.

Options

Options are essential financial instruments granting investors the right, without the obligation, to buy (call option) or sell (put option) an underlying asset at a predetermined price (strike price) within a specified timeframe (expiration date). They serve pivotal roles in portfolio management, risk hedging, and speculative strategies. Offering flexibility, options enable market participants to capitalize on both upward (long call) and downward (long put) price movements in the underlying asset, while also facilitating protection against adverse price fluctuations (short call/put) [44, 45, 46, 47].

While the majority of options fall into European or American categories, numerous other types exist, including Barrier options, Bermudan options, Asian options, and Lookback options, and in this project we focus on Asian options. These derivatives have been priced using various models such as the Black-Scholes-Merton model [48], Monte Carlo simulation [49, 50], and Perturbation methods [51]. Here, our aim is to derive analytic solutions for the Arithmetic Asian Options model described by a two state PDE (1.1), adapted from [52].

$$-rv + v_t + rxv_x + \frac{1}{2}\sigma^2 x^2 v_{xx} + xv_y = 0 \tag{1.1}$$

where $v = v(t, x, y)$. i.e, the dependent variable is v, and the independent variables are t, x, y, subject to the following boundary conditions:

$$v(T, x, y) = h(y), x \geq 0, \quad y \in \mathbb{R}, \tag{1.2}$$

$$v(t, 0, y) = e^{-r(T-t)} h(y), 0 \leq t \leq T, \quad y \in \mathbb{R}. \tag{1.3}$$

1.3 Aim and Objectives

The primary aim of this paper is to obtain an analytic solution to the PDE given in (1.1) using Lie symmetry methods. This goal will be accomplished through the following objectives:

1. Find the determining equations.
2. Solve for infinitesimal generators.
3. Find invariant solutions.
4. Investigate how changes in parameters such as volatility, interest rates, and time influence option prices.

Chapter 2

Methods

2.1 Symmetries of Differential Equations

2.1.1 Introduction

Many mathematical models have symmetries, especially those that are expressed in terms of DEs [53, 54, 55, 56]. Lie group theory—named after the Norwegian mathematician Marius Sophus Lie—is the branch of mathematics that represents and synthesizes the symmetries of DEs [57, 58, 59]. An organised method for finding the exact solutions to PDEs and ODEs is the Lie approach [60, 61, 62, 63, 64, 65, 66].

2.1.2 Preliminaries

In this section, we give some important definitions adopted from [67, 68, 69].

Definition 2.1.2.1. A k^{th}-order ($k \geq 1$) system E of s DEs is defined by

$$E^{\sigma}(x, u, u^{(1)}, \ldots, u^{(k)}) = 0, \quad \sigma = 1, \ldots, s, \tag{2.1}$$

where $u \equiv (u^1, u^2, \ldots, u^q)$ is the dependent vector, $x \equiv (x^1, x^2, \ldots, x^n)$ is the independent vector, and $u^{(1)}, u^{(2)}, \ldots, u^{(k)}$ are respectively the collection of all first, second, up to kth-order derivatives.

Definition 2.1.2.2. A symmetry transformation of the system (2.1) is an invertible transformation of the variables x and u, namely

$$\bar{x}^i = f^i(x,u), \quad \bar{u}^\alpha = \phi^\alpha(x,u), \quad i = 1,\ldots,n; \quad \alpha = 1,\ldots,q, \tag{2.2}$$

that leaves (2.1) form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^\sigma(\bar{x},\bar{u},\bar{u}^{(1)},\ldots,\bar{u}^{(k)}) = 0, \quad \sigma = 1,\ldots,s, \tag{2.3}$$

whenever (2.1) is satisfied.

Definition 2.1.2.3. A set G of transformations

$$T_a : \bar{x}^i = f^i(x,u,a), \quad \bar{u}^\alpha = \phi^\alpha(x,u,a), \quad i = 1,\ldots,n; \quad \alpha = 1,\ldots,q, \tag{2.4}$$

is called a continuous one-parameter (local) Lie-group of transformations in $\mathbb{R}^{n+q}$ provided the group properties of closure, identity, and inverses are satisfied. Here f^i and ϕ^α are differentiable functions and a is a real parameter which continuously takes values in a neighborhood $D \subseteq \mathbb{R}$ of $a = 0$.

Definition 2.1.2.4. An infinitesimal generator X of the group transformations G (2.4) is the differential operator of the form

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{2.5}$$

such that

$$\begin{aligned} \bar{x}^i &= x^i + a\xi^i(x,u) + O(a^2) = (1+aX)x^i, \\ \bar{u}^\alpha &= u^\alpha + a\eta^\alpha(x,u) + O(a^2) = (1+aX)u^\alpha. \end{aligned} \tag{2.6}$$

Here and throughout this section, the Einstein summation convention is adopted. The one-parameter group elements (2.6) are known as the *infinitesimal transformations* obtained from (2.4) by first-order (Taylor expansion) approximations around parameter $a = 0$. The operator (2.5) is also called the Lie point symmetry generator.

Definition 2.1.2.5. The extended infinitesimal generator $X^{[k]}$ of the kth prolonged (extended) group $G^{[k]}$ on the space $(x, u, \ldots, u^{(k)})$ is called the k^{th} prolongation of X, given by

$$\begin{aligned} X^{[k]} = {} & \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u^{(1)})\frac{\partial}{\partial u_i^\alpha} \\ & + \cdots + \zeta^\alpha_{(i_1 \ldots i_k)}\frac{\partial}{\partial u^\alpha_{(i_1\ldots i_k)}}. \end{aligned} \tag{2.7}$$

The coefficients ζ_s are defined recursively by the prolongation formulae

$$\begin{aligned} \zeta_i^\alpha &= D_i(\eta^\alpha) - u^\alpha_{(j)} D_i(\xi^j), \\ \zeta_\alpha^{ij} &= D_j(\zeta_i^\alpha) - u^\alpha_{(il)} D_j(\xi^l), \\ &\vdots \\ \zeta^\alpha_{i_1\ldots i_k} &= D_{i_k}(\zeta^\alpha_{i_1\ldots i_{k-1}}) - u^\alpha_{(i_1\ldots i_k l)} D_{i_k}(\xi^l), \end{aligned} \tag{2.8}$$

where

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_{(j)}\frac{\partial}{\partial u^\alpha} + u^\alpha_{(jk)}\frac{\partial}{\partial u^\alpha_{(k)}} \ldots ; u^\alpha_{(j)} = D_j(u^\alpha), u^\alpha_{(jk)} = D_j(u^\alpha_{(k)} \tag{2.9}$$

is the total derivative operator with respect to x^i.

2.1.3 Lie's Algorithm

In this section we introduce the necessary steps to be followed when calculating point symmetries of DEs, adopted from [70].

1. Write E given by (2.1) (the PDE or ODE being solved, such as (1.1)) such that all the terms are on the left-hand side.

2. Write the generator of symmetry

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{2.10}$$

3. Prolong the symmetry generator X to the order which is the same as that of E, i.e.,

$$\begin{aligned} X^{[k]} &= \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x, u, u^{(1)})\frac{\partial}{\partial u_i^\alpha} \\ &+ \cdots + \zeta^\alpha_{(i_1 \dots i_k)}\frac{\partial}{\partial u^\alpha_{(i_1 \dots i_k)}}. \end{aligned} \tag{2.11}$$

   where the variables ζ_i^α are given by (2.8).

4. Apply the prolonged generator $X^{[k]}$ on E evaluated on the surface (2.1) yielding the symmetry conditions

$$X^{[k]}\left(E^\sigma(x, u, u_{(1)}, \dots, u_{(k)})\right)|_{(2.1)} = 0, \quad \sigma = 1, \dots, s. \tag{2.12}$$

5. Substitute the ζ_i^α upon expansion of the symmetry conditions and replace the derivatives which are to be eliminated.

6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDEs in terms of ξ^i and η^α.

7. Solve the over-determined system for the infinitesimals ξ_i and η_α to obtain symmetries of E.

8. Construct one-parameter groups.

2.1.4 Lie Algebra

This section's contents are adopted from [60].

Definition 2.1.4.1. A Lie algebra is a vector space L over a field $\mathbb{F}$ with a binary operation $[-,-] : L \times L \to L$ called Lie bracket (also known as commutator), such that the following axioms are satisfied:

(i) Bilinearity: If $X_1, X_2, X_3 \in L$ and $a, b \in \mathbb{F}$, then

$$[aX_1 + bX_2, X_3] = a[X_1, X_3] + b[X_2, X_3].$$

(ii) Skew-Symmetry: If $X_1 \in L$, then

$$[X_1, X_1] = 0,$$

and this implies that, for all $X_1, X_2 \in L$,

$$[X_1, X_2] = -[X_2, X_1].$$

(iii) Jacobi Identity: If $X_1, X_2, X_3 \in L$, then

$$[X_1, [X_2, X_3]] + [X_2, [X_3, X_1]] + [X_3, [X_1, X_2]] = 0.$$

Definition 2.1.4.2. Consider a Lie algebra L. If the vector space L is finite-dimensional, its dimension is the dimension of the Lie algebra, that is, the finite-dimensional Lie algebra of dimension r is denoted by L_r.

In this project we take $\mathbb{F}$ to be the field of real numbers $\mathbb{R}$. We define the Lie Bracket $[-,-]$ on the set of vector field $\mathcal{V}$ as

$$[X_1, X_2] = X_1X_2 - X_2X_1 \quad \text{for any} \quad X_1, X_2 \in \mathcal{V}, \tag{2.13}$$

where

$$X_1 = \xi_1^i(x,u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{2.14}$$

and

$$X_2 = \xi_2^i(x,u)\frac{\partial}{\partial x^i} + \eta_2^\alpha(x,u)\frac{\partial}{\partial u^\alpha}. \tag{2.15}$$

The binary operation (2.13) makes the space of vector field $\mathcal{V}$ a Lie algebra.

2.1.5 Invariant Solutions

This section is adapted from [70]. The primary motivation for determining the symmetries of DEs is to use them to uncover the structure of the solution space. A notable characteristic of a symmetry is its ability to transform a solution into a different one. When dealing with PDEs, finding a general solution can be extremely challenging or even impossible. Therefore, it is often necessary to rely on particular solutions. Among these, invariant solutions can be systematically identified when the symmetries of the underlying equation are known.

Definition 2.1.5.1. A solution $u^\alpha = F^\alpha(x^1, x^2, \ldots, x^n)$ of E^α is invariant under the one-parameter group of transformations if

$$\bar{u}^\alpha = F^\alpha(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n). \tag{2.16}$$

If we perform the first-order Taylor expansion of (2.16) around $a = 0$ and use the first-order approximations of the one-parameter group we obtain

$$u^\alpha + a\eta^\alpha + \cdots = F^\alpha + aXF^\alpha + \cdots \tag{2.17}$$

whenever $u = F$. This equation implies that

$$\eta^\alpha = XF^\alpha \tag{2.18}$$

whenever $u = F$. Therefore,

$$u = F(x^1, \ldots, x^n), \tag{2.19}$$

is invariant under transformations provided

$$X(u^\alpha - F^\alpha)\big|_{(2.19)} = 0. \tag{2.20}$$

Conversely, it can be shown that if (2.20) is satisfied, then $u = F$ is invariant. Equation (2.20) represents a system of first-order quasi-linear PDEs that can be solved to obtain the functional form of F. Once this form is obtained, it is substituted back into the original system, leading to PDEs with fewer independent variables. If the symmetries of these reduced equations are known, further reductions can be performed. In favorable cases, this process results in closed-form solutions of the original system.

2.2 Application of Lie Symmetry Method in the current study

The infinitesimal generator for (1.1) w.r.t (2.5) is given by

$$X = \xi^1(t,x,y,v)\,\partial_t + \xi^2(t,x,y,v)\,\partial_x + \xi^3(t,x,y,v)\,\partial_y + \eta(t,x,y,v)\,\partial_v. \tag{2.21}$$

The second prolongation using (2.7) is given by

$$X^{[2]} = \xi^2\partial_x + \eta\partial_v + \zeta_t\partial_{v_t} + \zeta_x\partial_{v_x} + \zeta_y\partial_{v_y} + \zeta_{xx}\partial_{v_{xx}}. \tag{2.22}$$

Equation (2.21) is the infinitesimal generator of (1.1) if and only if

$$X^{[2]}\left(-rv + v_t + rxv_x + \frac{1}{2}\sigma^2x^2v_{xx} + xv_y\right)|_{(1.1)} = 0. \tag{2.23}$$

From (2.22) and (2.23), we have that

$$\xi^2\left(rv_x + \sigma^2 x v_{xx} + v_y\right) - r\eta + \zeta_t + rx\zeta_x + x\zeta_y + \frac{1}{2}\sigma^2 x^2 \zeta_{xx} = 0, \tag{2.24}$$

which simplifies to

$$2\xi^2\left(rv_x + \sigma^2 x v_{xx} + v_y\right) - 2r\eta + 2\zeta_t + 2rx\zeta_x + 2x\zeta_y + \sigma^2 x^2 \zeta_{xx} = 0. \tag{2.25}$$

The total derivatives w.r.t t, x, y are given as follows, respectively (With the use of (2.9)):

$$D_t = \frac{\partial}{\partial_t} + v_t\frac{\partial}{\partial_v} + v_{tt}\frac{\partial}{\partial_{v_t}} + v_{tx}\frac{\partial}{\partial_{v_x}} + v_{ty}\frac{\partial}{\partial_{v_y}} + ..., \tag{2.26}$$

$$D_x = \frac{\partial}{\partial_x} + v_x\frac{\partial}{\partial_v} + v_{tx}\frac{\partial}{\partial_{v_t}} + v_{xx}\frac{\partial}{\partial_{v_x}} + v_{xy}\frac{\partial}{\partial_{v_y}} + ..., \tag{2.27}$$

$$D_y = \frac{\partial}{\partial_y} + v_y\frac{\partial}{\partial_v} + v_{ty}\frac{\partial}{\partial_{v_t}} + v_{xy}\frac{\partial}{\partial_{v_x}} + v_{yy}\frac{\partial}{\partial_{v_y}} + ... \tag{2.28}$$

When considering (2.8), the ζ_s are given as follows:

$$\zeta_t = D_t(\eta) - v_t D_t(\xi^1) - v_x D_t(\xi^2) - v_y D_t(\xi^3), \tag{2.29}$$

$$\zeta_x = D_x(\eta) - v_t D_x(\xi^1) - v_x D_x(\xi^2) - v_y D_x(\xi^3), \tag{2.30}$$

$$\zeta_y = D_y(\eta) - v_t D_y(\xi^1) - v_x D_y(\xi^2) - v_y D_y(\xi^3), \tag{2.31}$$

$$\zeta_{xx} = D_x(\zeta_x) - v_{tx} D_x(\xi^1) - v_{xx} D_x(\xi^2) - v_{xy} D_x(\xi^3). \tag{2.32}$$

Now, from (2.26) and (2.29), we have that

$$\zeta_t = \eta_t + v_t(\eta_v - \xi^1_t) - v_t^2\xi^1_v - v_x\xi^2_t - v_tv_x\xi^2_v - v_y\xi^3_t - v_tv_y\xi^3_v. \tag{2.33}$$

From (2.27) and (2.30) we have that

$$\zeta_x = \eta_x + v_x(\eta_v - \xi^2_x) - v_x^2\xi^2_v - v_t\xi^1_x - v_tv_x\xi^1_v - v_y\xi^3_x - v_xv_y\xi^3_v. \tag{2.34}$$

From (2.28) and (2.31) we have that

$$\zeta_y = \eta_y + v_y(\eta_v - \xi^3_y) - v_y^2\xi^3_v - v_t\xi^1_y - v_tv_y\xi^1_v - v_x\xi^2_y - v_xv_y\xi^2_v. \tag{2.35}$$

From (2.27), (2.34), and (2.32) we have that

$$\begin{aligned}\zeta_{xx} = \eta_{xx} &+ v_x\eta_{vx} + v_{xx}\eta_v - v_{xx}\xi^2_x + v_x\eta_{vx} + v_x^2\eta_{vv} \\ &- v_x\xi^2_{xx} - v_x^2\xi^2_{vx} - v_x^2\xi^2_{vx} - v_x^3\xi^2_{vv} - 2v_xv_{xx}\xi^2_v - v_{tx}\xi^1_x \\ &- v_t\xi^1_{xx} - v_tv_x\xi^1_{vx} - v_tv_x\xi^1_{vx} - v_tv_x^2\xi^1_{vv} - v_tv_{xx}\xi^1_v - v_xv_{tx}\xi^1_v \\ &- v_xv_y\xi^3_{vx} - -v_x^2v_y\xi^3_{vv} - v_xv_{xy}\xi^3_v - v_yv_{xx}\xi^3_v - v_y\xi^3_{xx} - v_xv_y\xi^3_{vx} \\ &- v_{xy}\xi^3_x - v_{tx}\xi^1_x - v_xv_{tx}\xi^1_v - v_{xx}\xi^2_x - v_xv_{xx}\xi^2_v - v_{xy}\xi^3_x - v_xv_{xy}\xi^3_v,\end{aligned} \tag{2.36}$$

which simplifies to

$$\begin{aligned}\zeta_{xx} = \eta_{xx} &+ 2v_x\eta_{vx} + v_{xx}\eta_v - 2v_{xx}\xi^2_x + v_x^2\eta_{vv} - v_x\xi^2_{xx} \\ &- 2v_x^2\xi^2_{vx} - v_x^3\xi^2_{vv} - 3v_xv_{xx}\xi^2_v - 2v_{tx}\xi^1_x - v_t\xi^1_{xx} \\ &- 2v_tv_x\xi^1_{vx} - v_tv_x^2\xi^1_{vv} - v_tv_{xx}\xi^1_v - 2v_xv_{tx}\xi^1_v - 2v_xv_y\xi^3_{vx} \\ &- v_x^2v_y\xi^3_{vv} - 2v_xv_{xy}\xi^3_v - v_yv_{xx}\xi^3_v - v_y\xi^3_{xx} - 2v_{xy}\xi^3_x.\end{aligned} \tag{2.37}$$

Substituting (2.33), (2.34), (2.35), and (2.37) into (2.25), we have that

$$
\begin{aligned}
&2\xi^2(rv_x + \sigma^2 x v_{xx} + v_y) - 2r\eta + 2(\eta_t + v_t(\eta_v - \xi^1_t) - v_t^2\xi^1_v - v_x\xi^2_t \\
&- v_t v_x \xi^2_v - v_y\xi^3_t - v_t v_y \xi^3_v) + 2rx(\eta_x + v_x(\eta_v - \xi^2_x) - v_x^2\xi^2_v - v_t\xi^1_x \\
&- v_t v_x \xi^1_v - v_y \xi^3_x - v_x v_y \xi^3_v) + 2x(\eta_y + v_y(\eta_v - \xi^3_y) - v_y^2\xi^3_v - v_t\xi^1_y \\
&- v_t v_y \xi^1_v - v_x\xi^2_y - v_x v_y \xi^2_v) + \sigma^2 x^2(\eta_{xx} + 2v_x\eta_{vx} + v_{xx}\eta_v - 2v_{xx}\xi^2_x \\
&+ v_x^2\eta_{vv} - v_x\xi^2_{xx} - 2v_x^2\xi^2_{vx} - v_x^3\xi^2_{vv} - 3v_x v_{xx}\xi^2_v - 2v_{tx}\xi^1_x - v_t\xi^1_{xx} \\
&- 2v_t v_x \xi^1_{vx} - v_t v_x^2 \xi^1_{vv} - v_t v_{xx}\xi^1_v - 2v_x v_{tx}\xi^1_v - 2v_x v_y \xi^3_{vx} - v_x^2 v_y \xi^3_{vv} \\
&- 2v_x v_{xy}\xi^3_v - v_y v_{xx}\xi^3_v - v_y\xi^3_{xx} - 2v_{xy}\xi^3_x \,) = 0.
\end{aligned}
\tag{2.38}
$$

Now, from (1.1) we can have that

$$
v_t = rv - rxv_x - \frac{1}{2}\sigma^2 x^2 v_{xx} - xv_y \tag{2.39}
$$

When considering (2.39), (2.38) becomes

$$
\begin{aligned}
&2\xi^2 rv_x + 2\xi^2\sigma^2 xv_{xx} + 2\xi^2 v_y - 2r\eta + 2\eta_t + 2(rv - rxv_x \\
&- \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)(\eta_v - \xi^1_t) - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
&- xv_y)^2\xi^1_v - 2v_x\xi^2_t - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi^2_v \\
&- 2v_y\xi^3_t - 2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_y\xi^3_v + 2rx\eta_x \\
&+ 2rxv_x(\eta_v - \xi^2_x) - 2rxv_x^2\xi^2_v - 2rx(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
&- xv_y)\xi^1_x - 2rx(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi^1_v - 2rxv_y\xi^3_x \\
&- 2rxv_xv_y\xi^3_v + 2x\eta_y + 2xv_y(\eta_v - \xi^3_y) - 2xv_y^2\xi^3_v - 2x(rv - rxv_x \\
&- \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)\xi^1_y - 2x(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_y\xi^1_v \\
&- 2xv_x\xi^2_y - 2xv_xv_y\xi^2_v + \sigma^2x^2\eta_{xx} + 2\sigma^2x^2v_x\eta_{vx} + \sigma^2x^2v_{xx}\eta_v \\
&- 2\sigma^2x^2v_{xx}\xi^2_x + \sigma^2x^2v_x^2\eta_{vv} - \sigma^2x^2v_x\xi^2_{xx} - 2\sigma^2x^2v_x^2\xi^2_{vx} - \sigma^2x^2v_x^3\xi^2_{vv} \\
&- 3\sigma^2x^2v_xv_{xx}\xi^2_v - 2\sigma^2x^2v_{tx}\xi^1_x - \sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} \\
&- xv_y)\xi^1_{xx} - 2\sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x\xi^1_{vx} \\
&- \sigma^2x^2(rv - rxv_x - \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_x^2\xi^1_{vv} - \sigma^2x^2(rv - rxv_x \\
&- \frac{1}{2}\sigma^2x^2v_{xx} - xv_y)v_{xx}\xi^1_v - 2\sigma^2x^2v_xv_{tx}\xi^1_v - 2\sigma^2x^2v_xv_y\xi^3_{vx} \\
&- \sigma^2x^2v_x^2v_y\xi^3_{vv} - 2\sigma^2x^2v_xv_{xy}\xi^3_v - \sigma^2x^2v_yv_{xx}\xi^3_v - \sigma^2x^2v_y\xi^3_{xx} \\
&- 2\sigma^2x^2v_{xy}\xi^3_x = 0,
\end{aligned}
\tag{2.40}
$$

which simplifies to

$$\begin{aligned}
&2\xi^2 r v_x + 2\xi^2\sigma^2 x v_{xx} + 2\xi^2 v_y - 2r\eta + 2\eta_t + 2rv\eta_v - 2rxv_x\eta_v \\
&- \sigma^2 x^2 v_{xx}\eta_v - 2xv_y\eta_v - 2rv\xi^1_t + 2rxv_x\xi^1_t + \sigma^2x^2v_{xx}\xi^1_t + 2xv_y\xi^1_t \\
&- 2r^2v^2\xi^1_v + 4vr^2xv_x\xi^1_v + 4xvrv_y\xi^1_v - 2r^2x^2v_x^2\xi^1_v - 4rx^2v_xv_y\xi^1_v \\
&- 2x^2v_y^2\xi^1_v - 2rx^3\sigma^2v_xv_{xx}\xi^1_v - 2\sigma^2x^3v_yv_{xx}\xi^1_v - \frac{1}{2}\sigma^4x^4v_{xx}^2\xi^1_v \\
&+ 2rv\sigma^2x^2v_{xx}\xi^1_v - 2v_x\xi^2_t - 2rvv_x\xi^2_v + 2rxv_x^2\xi^2_v + \sigma^2x^2v_{xx}v_x\xi^2_v \\
&+ 2xv_yv_x\xi^2_v - 2v_y\xi^3_t - 2rvv_y\xi^3_v + 2rxv_xv_y\xi^3_v + \sigma^2x^2v_{xx}v_y\xi^3_v \\
&+ 2xv_y^2\xi^3_v + 2rx\eta_x + 2rxv_x\eta_v - 2rxv_x\xi^2_x - 2rxv_x^2\xi^2_v - 2r^2xv\xi^1_x \\
&+ 2r^2x^2v_x\xi^1_x + r\sigma^2x^3v_{xx}\xi^1_x + 2rx^2v_y\xi^1_x - 2r^2xvv_x\xi^1_v + 2r^2x^2v_x^2\xi^1_v \\
&+ r\sigma^2x^3v_{xx}v_x\xi^1_v + 2rx^2v_yv_x\xi^1_v - 2rxv_y\xi^3_x - 2rxv_xv_y\xi^3_v + 2x\eta_y \\
&+ 2xv_y\eta_v - 2xv_y\xi^3_y - 2xv_y^2\xi^3_v - 2xrv\xi^1_y + 2x^2rv_x\xi^1_y + \sigma^2x^3v_{xx}\xi^1_y \\
&+ 2x^2v_y\xi^1_y - 2xrvv_y\xi^1_v + 2x^2rv_xv_y\xi^1_v + \sigma^2x^3v_{xx}v_y\xi^1_v + 2x^2v_y^2\xi^1_v \\
&- 2xv_x\xi^2_y - 2xv_xv_y\xi^2_v + \sigma^2x^2\eta_{xx} + 2\sigma^2x^2v_x\eta_{vx} + \sigma^2x^2v_{xx}\eta_v \\
&- 2\sigma^2x^2v_{xx}\xi^2_x + \sigma^2x^2v_x^2\eta_{vv} - \sigma^2x^2v_x\xi^2_{xx} - 2\sigma^2x^2v_x^2\xi^2_{vx} - \sigma^2x^2v_x^3\xi^2_{vv} \\
&- 3\sigma^2x^2v_xv_{xx}\xi^2_v - 2\sigma^2x^2v_{tx}\xi^1_x - \sigma^2x^2rv\xi^1_{xx} + \sigma^2x^3rv_x\xi^1_{xx} \\
&+ \frac{1}{2}\sigma^4x^4v_{xx}\xi^1_{xx} + \sigma^2x^3v_y\xi^1_{xx} - 2\sigma^2x^2rvv_x\xi^1_{vx} + 2\sigma^2x^3rv_x^2\xi^1_{vx} \\
&+ \sigma^4x^4v_{xx}v_x\xi^1_{vx} + 2\sigma^2x^3v_yv_x\xi^1_{vx} - \sigma^2x^2rvv_x^2\xi^1_{vv} + \sigma^2x^3rv_x^3\xi^1_{vv} \\
&+ \frac{1}{2}\sigma^4x^4v_{xx}v_x^2\xi^1_{vv} + \sigma^2x^3v_yv_x^2\xi^1_{vv} - \sigma^2x^2rvv_{xx}\xi^1_v + \sigma^2x^3rv_xv_{xx}\xi^1_v \\
&+ \frac{1}{2}\sigma^4x^4v_{xx}^2\xi^1_v + \sigma^2x^3xv_yv_{xx}\xi^1_v - 2\sigma^2x^2v_xv_{tx}\xi^1_v - 2\sigma^2x^2v_xv_y\xi^3_{vx} \\
&- \sigma^2x^2v_x^2v_y\xi^3_{vv} - 2\sigma^2x^2v_xv_{xy}\xi^3_v - \sigma^2x^2v_yv_{xx}\xi^3_v - \sigma^2x^2v_y\xi^3_{xx} \\
&- 2\sigma^2x^2v_{xy}\xi^3_x = 0.
\end{aligned} \tag{2.41}$$

Now, separating (2.41) w.r.t powers and products of v we get the following determining equations:

$$v_{tx} : \xi^1_x = 0, \tag{2.42}$$

$$v_x v_{xy} : \xi^3_v = 0, \tag{2.43}$$

$$v_{xy} : \xi^3_x = 0, \tag{2.44}$$

$$v_x v_{tx} : \xi^1_v = 0, \tag{2.45}$$

$$v_{xx} v_x^2 : \xi^1_{vv} = 0, \tag{2.46}$$

$$v_y v_x^2 : x\xi^1_{vv} - \xi^3_{vv} = 0, \tag{2.47}$$

$$v_x v_y : -2\xi^3_{vx} + 2x\xi^1_{vx} = 0, \tag{2.48}$$

$$v_x^3 : -\xi^2_{vv} + 2rx\xi^1_{vv} = 0, \tag{2.49}$$

$$v_x v_{xx} : -\xi^2_v + \sigma^2 x^2 \xi^1_{vx} = 0, \tag{2.50}$$

$$v_x^2 : \eta_{vv} - 2\xi^2_{vx} - 2rx\xi^1_{vx} - rv\xi^1_{vv} = 0, \tag{2.51}$$

$$v_{xx} : 4\xi^2 + 2x\xi^1_t + 2rvx\xi^1_v + 2rx^2\xi^1_x + 2x^2\xi^1_y - 4x\xi^2_x + \sigma^2 x^3 \xi^1_{xx} = 0, \tag{2.52}$$

$$v_y : 2\xi^2 + 2x\xi^1_t + 2rvx\xi^1_v - 2\xi^3_t - 2rv\xi^3_v + 2rx^2\xi^1_x - 2rx\xi^3_x + \sigma^2 x^3 \xi^1_{xx} - \sigma^2 x^2 \xi^3_{xx} - 2x\xi^3_y + 2x^2\xi^1_y = 0, \tag{2.53}$$

$$v_x : 2\xi^2 r + 2rx\xi^1_t - 2r^2xv\xi^1_v - 2\xi^2_t - 2rv\xi^2_v - 2rx\xi^2_x + 2r^2x^2\xi^1_x + 2rx^2\xi^1_y - 2x\xi^2_y + 2\sigma^2x^2\eta_{vx} - r\sigma^2x^3\xi^1_{xx} - 2rv\sigma^2x^2\xi^1_{vx} = 0, \tag{2.54}$$

$$\text{Constant-terms:} - 2r\eta + 2\eta_t + 2rv\eta_v - 2rv\xi^1_t + 2rx\eta_x - 2r^2xv\xi^1_x - 2rvx\xi^1_y + \sigma^2x^2\eta_{xx} - rv\sigma^2x^2\xi^1_{xx} - 2r^2v^2\xi^1_v + 2x\eta_y = 0. \tag{2.55}$$

2.2.1 Solving the determining equations

From (2.42), we have that

$$\xi^1 = \xi^1(t, y, v). \tag{2.56}$$

From (2.45), we have that

$$\xi^1 = \xi^1(t, x, y). \tag{2.57}$$

Now, from (2.56) and (2.57) we have that

$$\xi^1 = \xi^1(t, y). \tag{2.58}$$

When considering (2.43), we have that

$$\xi^3 = \xi^3(t, x, y), \tag{2.59}$$

and from (2.44) we have that

$$\xi^3 = \xi^3(t, y, v). \tag{2.60}$$

Using (2.60) and (2.59) we have that

$$\xi^3 = \xi^3(t, y). \tag{2.61}$$

From (2.46) and (2.47), we have that

$$\begin{aligned} &\xi^3_{vv} = 0, \\ &\text{which is identically satisfied by (2.61).} \end{aligned} \tag{2.62}$$

By (2.48), (2.58), and (2.61), equation (2.48) is satisfied.
Taking (2.45), (2.58), and (2.50), we have that

$$\xi^2 = \xi^2(t, x, y). \tag{2.63}$$

Again, when taking (2.46) and (2.49), we have that

$$\begin{aligned} &\xi^2_{vv} = 0, \\ &\text{which is identically satisfied by (2.63).} \end{aligned} \tag{2.64}$$

Considering (2.51), (2.45), (2.46), (2.58), and (2.63), we have that

$$\begin{aligned} &\eta_{vv} = 0, \\ &\Rightarrow \eta = a(t, x, y)v + b(t, x, y). \end{aligned} \tag{2.65}$$

Equations (2.53), (2.58), and (2.61) yield

$$x\xi^1_t - \xi^3_t + x^2\xi^1_y - x\xi^3_y + \xi^2 = 0, \tag{2.66}$$

which when differentiated w.r.t x thrice gives

$$\xi^2(t,x,y) = c(t,y) + xd(t,y) + x^2 e(t,y). \tag{2.67}$$

Substituting (2.67) into (2.66) and separating w.r.t powers of x we have that

$$\begin{aligned} c(t,y) &= \xi^3_t, \\ d(t,y) &= \xi^3_y - \xi^1_t, \\ e(t,y) &= -\xi^1_y. \end{aligned} \tag{2.68}$$

Then, from (2.67) and (2.68) we have that

$$\xi^2(t,x,y) = \xi^3_t + x\xi^3_y - x\xi^1_t - x^2\xi^1_y. \tag{2.69}$$

Now, from (2.52), (2.42), (2.58), (2.65) we have that

$$2\xi^2 + x^2\xi^1_y + x\xi^1_t - 2x\xi^2_x = 0. \tag{2.70}$$

Using (2.69) and (2.70), we have that

$$\begin{aligned} &2(\xi^3_t + x\xi^3_y - x\xi^1_t - x^2\xi^1_y) + x^2\xi^1_y \\ &+ x\xi^1_t - 2x(-2x\xi^1_y + \xi^3_y - \xi^1_t) \quad = 0 \\ &\Rightarrow 3x^2\xi^1_y + x\xi^1_t + 2\xi^3_t = 0. \end{aligned} \tag{2.71}$$

Differentiating (2.71) w.r.t x twice we have that

$$\begin{aligned} &18\xi^1_y = 0, \\ &\Rightarrow \xi^1 = \xi^1(t). \end{aligned} \tag{2.72}$$

And thus, (2.71) and (2.72) yield

$$x\xi^1_t + 2\xi^3_t = 0, \tag{2.73}$$

which when differentiated w.r.t x gives

$$\begin{aligned} &\xi^1_t = 0, \\ &\Rightarrow \xi^1 = K_1. \end{aligned} \tag{2.74}$$

Thus, from (2.73) and (2.74), we have that

$$\begin{aligned} &\xi^3_t = 0, \\ &\Rightarrow \xi^3 = \xi^3(y). \end{aligned} \tag{2.75}$$

When considering (2.54), (2.58), and (2.61), we have that

$$2rx^2\xi^1_y - 2x\xi^2_y - 2rx\xi^2_x + 2\sigma^2x^2\eta_{vx} - \sigma^2x^2\xi^2_{xx} + 2rx\xi^1_t - 2\xi^1_t = 0. \tag{2.76}$$

Taking into consideration (2.69) (2.74), and (2.75), equation (2.76) becomes

$$-x\xi^3_{yy} - r\xi^3_y + x\sigma^2 a_x = 0. \tag{2.77}$$

Differentiating (2.77) w.r.t x gives and solving the output yields

$$a(t,x,y) = f(t,y) + \frac{x\xi^3_{yy}}{\sigma^2}. \tag{2.78}$$

When substituting (2.78) into (2.65) we have that

$$\eta = \left(f(t,y) + \frac{x\xi^3_{yy}}{\sigma^2}\right) v + b(t,x,y). \tag{2.79}$$

From (2.55) and (2.58), we have that

$$-2r\eta + 2rv\eta_v + 2\eta_t + 2rx\eta_x + x^2\sigma^2\eta_{xx} + 2x\eta_y - 2rv\left(x\xi^1_y + \xi^1_t\right) = 0, \tag{2.80}$$

which when considering (2.74) and (2.65), becomes

$$\begin{aligned} & vx(a_y) + x(b_y) + rvx(a_x) + rx(b_x) + \frac{1}{2}vx^2\sigma^2(a_{xx}) \\ & + \frac{1}{2}x^2\sigma^2(b_{xx}) + v(a_t) + b_t - rb = 0. \end{aligned} \tag{2.81}$$

When considering (2.78), equation (2.81) becomes

$$\begin{aligned} & \frac{rvx\xi^3_{yy}}{\sigma^2} + \frac{vx^2\xi^3_{yyy}}{\sigma^2} + vx(f_y) + v(f_t) \\ & + x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \end{aligned} \tag{2.82}$$

Differentiating (2.82) w.r.t v gives

$$f_t + xf_y + \frac{rx\xi^3_{yy}}{\sigma^2} + \frac{x^2\xi^3_{yyy}}{\sigma^2} = 0. \tag{2.83}$$

Again, differentiating (2.83) w.r.t x twice gives

$$\frac{2\xi^3_{yyy}}{\sigma^2} = 0, \tag{2.84}$$

which when solving yields

$$\xi^3 = K_2 + yK_3 + y^2K_4. \tag{2.85}$$

Now, when considering (2.82) and (2.85) we have that

$$\begin{aligned} & \frac{2rvxK_4}{\sigma^2} + vx(f_y) + v(f_t) + x(b_y) + rx(b_x) \\ & + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0, \end{aligned} \tag{2.86}$$

which when differentiated w.r.t v, then w.r.t x gives

$$\frac{2rK_4}{\sigma^2} + f_y = 0. \tag{2.87}$$

Solving for f on (2.87) yields

$$f = -\frac{2ryK_4}{\sigma^2} + g(t). \tag{2.88}$$

Now, from (2.86) and (2.88) we have that

$$v(g_t) + x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \tag{2.89}$$

Differentiating (2.89) w.r.t v gives

$$\begin{aligned} g_t &= 0, \\ \Rightarrow g &= K_5. \end{aligned} \tag{2.90}$$

Taking into consideration (2.90), equation (2.89) becomes

$$x(b_y) + rx(b_x) + \frac{1}{2}x^2\sigma^2(b_{xx}) + b_t - rb = 0. \tag{2.91}$$

Now, when considering (2.69), (2.74), and (2.85) we have that

$$\xi^2 = xK_3 + 2xyK_4. \tag{2.92}$$

Again, when considering (2.79), (2.85), (2.88), and (2.90) we have that

$$\eta = v\left(\frac{2xK_4 - 2ryK_4}{\sigma^2} + K_5\right) + b(t, x, y). \tag{2.93}$$

Substituting (2.74), (2.85), (2.92), and (2.93) into (2.21) gives

$$\begin{aligned} X = {} & K_1\partial_t + (xK_3 + 2xyK_4)\,\partial_x + \left(K_2 + yK_3 + y^2K_4\right)\partial_y \\ & + \left(v\left(\frac{2xK_4 - 2ryK_4}{\sigma^2} + K_5\right) + b(t, x, y)\right)\partial_v, \end{aligned}$$

where K_1, K_2, K_3, K_4 and K_5 are constants, and $b(t, x, y)$ is some function. (2.94)

Now, (2.94) will be used to find the infinitesimal generators for (1.1) as follows:

Case 1:

When $K_1 = 1, K_2 = K_3 = K_4 = K_5 = b(t, x, y) = 0$ we have

$$X_1 = \partial_t. \tag{2.95}$$

Case 2:

When $K_2 = 1, K_1 = K_3 = K_4 = K_5 = b(t, x, y) = 0$ we have

$$X_2 = \partial_y. \tag{2.96}$$

Case 3:

When $K_3 = 1, K_1 = K_2 = K_4 = K_5 = b(t, x, y) = 0$ we have

$$X_3 = x\partial_x + y\partial_y. \tag{2.97}$$

Case 4:

When $K_4 = 1, K_1 = K_2 = K_3 = K_5 = b(t, x, y) = 0$ we have

$$X_4 = 2xy\partial_x + y^2\partial_y + v\left(\frac{2x - 2ry}{\sigma^2}\right)\partial_v. \tag{2.98}$$

Case 5:

When $K_5 = 1, K_1 = K_2 = K_3 = K_4 = b(t, x, y) = 0$ we have

$$X_5 = v\partial_v. \tag{2.99}$$

Case 6:

When $K_1 = K_2 = K_3 = K_4 = K_5 = 0$ we have

$$X_b = b(t, x, y)\partial_v. \tag{2.100}$$

2.2.2 Finding the Lie Brackets

Let $b(t, x, y) = b$.

$[X_1, X_1]$:

$$[X_1, X_1] = 0 \tag{2.101}$$

$[X_1, X_2]$:

$$\begin{aligned} [X_1, X_2] &= X_1 X_2 - X_2 X_1 \\ &= \partial_t \partial_y - \partial_y \partial_t \\ &= 0 \end{aligned} \tag{2.102}$$

$[X_1, X_3]$:

$$\begin{aligned} [X_1, X_3] &= X_1 X_3 - X_3 X_1 \\ &= \partial_t (x\partial_x + y\partial_y) - (x\partial_x + y\partial_y)\partial_t \\ &= x(\partial_t \partial_x) + y(\partial_t \partial_y) - x(\partial_t \partial_x) - y(\partial_t \partial_y) \\ &= 0 \end{aligned} \tag{2.103}$$

$[X_2, X_1]$:

$$\begin{aligned} [X_2, X_1] &= -[X_1, X_2] \\ &= 0 \end{aligned} \tag{2.104}$$

$[X_2, X_2]$:

$$[X_2, X_2] = 0 \tag{2.105}$$

$[X_2, X_3]$:

$$\begin{aligned} [X_2, X_3] &= X_2 X_3 - X_3 X_2 \\ &= \partial_y (x\partial_x + y\partial_y) - (x\partial_x + y\partial_y)\partial_y \\ &= \partial_y \\ &= X_2 \end{aligned} \tag{2.106}$$

$[X_3, X_1]$**:**

$$\begin{aligned}[X_3, X_1] &= -[X_1, X_3] \\ &= 0\end{aligned} \tag{2.107}$$

$[X_3, X_2]$**:**

$$\begin{aligned}[X_3, X_2] &= -[X_2, X_3] \\ &= -X_2\end{aligned} \tag{2.108}$$

$[X_3, X_3]$**:**

$$[X_3, X_3] = 0 \tag{2.109}$$

$\nearrow [X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_b
X_1	0	0	0	0	0	$(\partial_t b)\partial_v$
X_2	0	0	X_2	$2X_3 - \frac{2r}{\sigma^2}X_5$	0	$(\partial_y b)\partial_v$
X_3	0	$-X_2$	0	$X''b$	0	Xb
X_4	0	$-(2X_3 - \frac{2r}{\sigma^2}X_5)$	$-X''b$	0	0	$X'b$
X_5	0	0	0	0	0	$-X_b$
X_b	$-(\partial_t b)\partial_v$	$-(\partial_y b)\partial_v$	$-Xb$	$-X'b$	X_b	0

Table 2.1: Table of Lie brackets

Where

$$\begin{aligned}Xb &= x(\partial_x b)\partial_v + y(\partial_y b)\partial_v, \\ X'b &= 2xy\partial_x b_v + y^2\partial_y b_v - b\left(\frac{2x - 2ry}{\sigma^2}\right)\partial_v, \\ X''b &= 2xy\partial_x + y^2\partial_y - \frac{2}{\sigma^2}X_5 - \frac{2r}{\sigma^2}X_5.\end{aligned} \tag{2.110}$$

2.2.3 Finding invariant solutions

To determine the form of the solution that is invariant under X, we need to solve the quasi-linear PDE given by (2.20).

If we consider the combination of (2.95) and (2.96) to form a new infinitesimal generator, say X_6, we have

$$X_6 = \partial_t + \partial_y, \tag{2.111}$$

which yields the following characteristic equation;

$$\frac{d_t}{1} = \frac{d_y}{1} = \frac{d_v}{0}. \tag{2.112}$$

Taking the first and second ratios of (2.112) we have that

$$\begin{aligned} &\frac{d_t}{1} = \frac{d_y}{1}, \\ &\Rightarrow t = y + C_1, \\ &\Rightarrow C_1 = t - y, \\ &\text{where } C_1 \text{ is a constant of integration.} \end{aligned} \tag{2.113}$$

Taking the third ratio of (2.112) we have

$$\begin{aligned} &v = C_2, \\ &\text{where } C_2 \text{ is a constant of integration.} \end{aligned} \tag{2.114}$$

From (2.113) and (2.114), we thus have

$$\begin{aligned} &v(t,x,y) = k(t-y) \text{ or } v(t,x,y) = k(C_1), \\ &\text{which is an invariant solution, where k is some arbitrary function.} \end{aligned} \tag{2.115}$$

Substituting (2.115) into (1.1) yields the following equation:

$$\begin{aligned} &- rk(t-y) + \frac{\partial}{\partial t}k(t-y) + rx\frac{\partial}{\partial x}k(t-y) \\ &+ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}k(t-y) + x\frac{\partial}{\partial y}k(t-y) = 0, \end{aligned} \tag{2.116}$$

which simplifies to

$$-rk + k' - xk' = 0, \tag{2.117}$$

which further simplifies to

$$-rk + k'(1-x) = 0. \tag{2.118}$$

When solving for k we have

$$\begin{aligned} &k = Me^{\left(\frac{r(t-y)}{1-x}\right)}, \ \ x \neq 1, \\ &\text{where } M \text{ is a constant of integration.} \end{aligned} \tag{2.119}$$

Using (2.115) and (2.119) we have the following invariant solution for (1.1):

$$v(t,x,y) = Me^{\left(\frac{r(t-y)}{1-x}\right)}. \tag{2.120}$$

If we take the combination of (2.95), (2.96), and (2.99) to form a new infinitesimal generator, say X_7, we have

$$X_7 = \partial_t + \partial_y + v\partial_v, \tag{2.121}$$

which yields the following characteristic equation

$$\frac{d_t}{1} = \frac{d_y}{1} = \frac{d_v}{v}. \tag{2.122}$$

Taking the first and second ratios of (2.122) we have that

$$\begin{aligned} &\frac{d_t}{1} = \frac{d_y}{1}, \\ &\Rightarrow t = y + C_3, \\ &\Rightarrow C_3 = t - y, \\ &\text{where } C_3 \text{ is a constant of integration.} \end{aligned} \tag{2.123}$$

Taking the first and third ratios of (2.122) we have

$$\begin{aligned} &\frac{d_t}{1} = \frac{d_v}{v}, \\ &\Rightarrow t = \ln v + C_4, \\ &\Rightarrow C_4 = t - \ln v, \\ &\text{where } C_4 \text{ is a constant of integration.} \end{aligned} \tag{2.124}$$

Now, from (2.123) and (2.124) we can deduce that

$$\begin{aligned} &t - \ln v = g(t - y) \\ &\Rightarrow v(t, x, y) = e^{t-g(t-y)} \text{ or } v(t, x, y) = e^{t-g(C_3)}, \\ &\text{which is an invariant solution, where } g \text{ is some arbitrary function.} \end{aligned} \tag{2.125}$$

Substituting (2.125) into (1.1) yields the following equation:

$$\begin{aligned} &- re^{t-g(t-y)} + \frac{\partial}{\partial t}e^{t-g(t-y)} + rx\frac{\partial}{\partial x}e^{t-g(t-y)} \\ &+ \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}e^{t-g(t-y)} + x\frac{\partial}{\partial y}e^{t-g(t-y)} = 0, \end{aligned} \tag{2.126}$$

which simplifies to

$$-re^{t-g} + e^{t-g}(1 - g') + xe^{t-g}g' = 0, \tag{2.127}$$

which further simplifies to

$$e^{t-g}(1 - r + (-1 + x)g') = 0. \tag{2.128}$$

Solving for g we have that

$$g = \frac{(t-y)(-1+r)}{-1+x} + N, \ \ x \neq 1, \tag{2.129}$$
where N is a constant of integration.

Then from (2.125) and (2.129) we have the following invariant solution for (1.1):

$$v(t, x, y) = e^{t - \frac{(t-y)(-1+r)}{-1+x} + N} \tag{2.130}$$

Considering (2.97) we have the following characteristic equation

$$\frac{d_x}{x} = \frac{d_y}{y} = \frac{d_v}{0}. \tag{2.131}$$

Taking the first and second ratios of (2.131) we have

$$\begin{aligned} &\frac{d_x}{x} = \frac{d_y}{y}, \\ &\Rightarrow lnx = lny + lnC_5, \\ &\Rightarrow C_5 = \frac{x}{y}, \ \ y \neq 0, \end{aligned} \tag{2.132}$$

where C_5 is a constant of integration.

Taking the third ratio of (2.131) we have

$$v = C_6, \tag{2.133}$$
where C_6 is a constant of integration.

From (2.132) and (2.133), we thus have

$$v(t,x,y) = f\left(\frac{x}{y}\right) \text{ or } v(t,x,y) = f(C_5), \tag{2.134}$$

which is an invariant solution, where f is some arbitrary function.

Now, substituting (2.134) into (1.1) we have that

$$\begin{aligned} & -rf\left(\frac{x}{y}\right) + \frac{\partial}{\partial t}f\left(\frac{x}{y}\right) + rx\frac{\partial}{\partial x}f\left(\frac{x}{y}\right) \\ & + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}f\left(\frac{x}{y}\right) + x\frac{\partial}{\partial y}f\left(\frac{x}{y}\right) = 0, \end{aligned} \tag{2.135}$$

which simplifies to

$$-rf + \frac{rx}{y}f' + \frac{\sigma^2 x^2}{2y^2}f'' - \frac{x^2}{y^2}f' = 0. \tag{2.136}$$

Considering (2.98) we have the following characteristic equations

$$\frac{d_x}{2xy} = \frac{d_y}{y^2} = \frac{\sigma^2}{v(2x-2ry)}d_v. \tag{2.137}$$

Taking the first and second ratios of (2.137) we have that

$$\begin{aligned} & \frac{d_x}{2xy} = \frac{d_y}{y^2}, \\ & \Rightarrow \frac{d_x}{x} = 2\frac{d_y}{y}, \\ & \Rightarrow \ln x = 2\ln y + \ln C_7, \\ & \Rightarrow C_7 = \frac{x}{y^2}, \ \ y^2 \neq 0, \end{aligned} \tag{2.138}$$

where C_7 is a constant of integration.

Taking the first and third ratios of (2.137) we have that

$$\begin{aligned}
&\frac{d_x}{2xy} = \frac{\sigma^2}{v(2x-2ry)} d_v, \\
&\Rightarrow \frac{d_x}{x} = \frac{y\sigma^2}{v(x-ry)} d_v, \\
&\Rightarrow \ln x + C_8 = \frac{y\sigma^2}{(x-ry)} \ln v, \\
&\Rightarrow C_8 = \frac{y\sigma^2}{(x-ry)} \ln v - \ln x, \;\; x \neq ry, \\
&\text{where } C_8 \text{ is a constant of integration.}
\end{aligned} \tag{2.139}$$

From (2.138) and (2.139) we have that

$$\begin{aligned}
&\frac{y\sigma^2}{(x-ry)} \ln v - \ln x = j\left(\frac{x}{y^2}\right), \\
&\Rightarrow \ln v = \left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right) + \left(\frac{x-ry}{y\sigma^2}\right) \ln x, \\
&\Rightarrow v(t,x,y) = e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right), \;\; y\sigma^2 \neq 0,
\end{aligned} \tag{2.140}$$

which is an invariant solution, where j is some arbitrary function.

Now, substituting (2.140) into (1.1) gives the following equation

$$\begin{aligned}
&-r\left(e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right)\right) \\
&+\frac{\partial}{\partial t}\left(e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right)\right) \\
&+rx\frac{\partial}{\partial x}\left(e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right)\right) \\
&+\frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2}\left(e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right)\right) \\
&+x\frac{\partial}{\partial y}\left(e^{\left(\frac{x-ry}{y\sigma^2}\right) j\left(\frac{x}{y^2}\right)} + \left(\frac{x-ry}{y\sigma^2}\right)\right) = 0,
\end{aligned} \tag{2.141}$$

which simplifies to

$$
\begin{aligned}
& - r\left(e^{\left(\frac{x-ry}{y\sigma^2}\right)j} + \left(\frac{x-ry}{y\sigma^2}\right)\right) \\
& + e^{\left(\frac{x-ry}{y\sigma^2}\right)j} x\left(-\left(\frac{x-ry}{y^2\sigma^2}\right)j - 2x\left(\frac{x-ry}{y^4\sigma^2}\right)j'\right) \\
& + rx\left(x^2ry\sigma^2 + e^{\left(\frac{x-ry}{y\sigma^2}\right)j}\left(\frac{j}{y\sigma^2}\right) + \left(\frac{x-ry}{y^3\sigma^2}\right)j'\right) \\
& + \frac{x^2}{y} + \frac{1}{2}x^2\sigma^2 e^{\left(\frac{x-ry}{y\sigma^2}\right)j}\left(\left(\frac{j}{y\sigma^2}\right) + \left(\frac{x-ry}{y^3\sigma^2}\right)j'\right)^2 \\
& + \frac{1}{2}x^2\sigma^2 e^{\left(\frac{x-ry}{y\sigma^2}\right)j}\left(\frac{2}{y^3\sigma^2}j'\right) + \left(\frac{x-ry}{y^5\sigma^2}\right)j'' = 0.
\end{aligned}
\tag{2.142}
$$

Chapter 3

Discussion and conclusion

In this research, we successfully found the determining equations, solved for the infinitesimal generators, and found four invariant solutions, some of which are expressed in terms of arbitrary functions. These achievements provide a robust foundation for understanding the symmetry properties and potential simplifications of the underlying mathematical model. The invariant solutions are particularly valuable as they offer flexibility and adaptability, allowing for further specification based on additional conditions or constraints, which is crucial in fields such as financial mathematics.

For future work, we shall focus on solving for the arbitrary functions f and j to find the final invariant solutions in a simplified form. Additionally, we will analyze the impact of different boundary conditions on option prices using these invariant solutions. Further investigation will be directed towards understanding how changes in parameters such as volatility, interest rates, and time influence option prices. Finally, we will compare and evaluate the accuracy and efficiency of the invariant solutions by plotting them against the exact solutions of the model.

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