NONLINEAR ACOUSTICS IN MEDICAL ULTRASOUND

A NUMERICAL APPROXIMATION OF THE WESTERVELT EQUATION USING FINITE DIFFERENCE METHOD

THEKO SEKHESA

A project submitted to the Faculty of Science and Technology, National University of Lesotho, in partial fulfilment of the requirements for the degree of Bachelor of Science Honours in Applied Mathematics

June 2023

DECLARATION

I, Theko Sekhesa, student number 201702343, declare that this project submitted for the degree of Bachelor of Science Honours in Applied Mathematics at National University of Lesotho has not been previously submitted by me at this or any other University. Further, I declare that this is my original work and any work done by others has been acknowledged in accordance.

____

(Initials. Surname)

____________day of _________________20 ________

ABSTRACT

In this work, an approximate solution for the 1-Dimensional Westervelt equation is found. This is done primarily using a numerical approximation method called Finite Difference Method. The first part of this work is to approximate the solution in regions of the domain where the nonlinearities are negligible. The second part of this work is to approximate a solution in regions of the domain where the nonlinearities are non-negligible.

DEDICATION

To my family and Friends

ACKNOWLEDGEMENTS

I would like to convey my heartfelt gratitude to my project supervisor Mr Ngaka John Nchejane, for his unwavering support and guidance. Secondly, to all my lecturers who went out of their way to assist in all challenges I faced. Lastly, to my family and friends, thank you for the emotional support.

Contents

INTRODUCTION

The application of nonlinear acoustics in the field of medicine has a rich history stretching back to nineteenth century. It has widely infiltrated both known pillers of the field being diagnostic medicine and surgical medicine. For a very long time, evasive surgery has been the number one go to practice as far as medical diagnosis was concerned. This involved cutting into the human body for a better image so as to make a more informed decision when diagnosing a patient.

But as new and complex ailments developed throughout the years, this became too much of a risk as patients would lose immense amounts of blood or develop fatal infections. These factors created an immediate need for an alternative imaging modality, thus, the era of ultrasonography amongst other imaging methods was rushed in [1]. Medical ultrasonography is the imaging of the human body using ultrasound. By ultrasound it is meant those waves that have frequencies that are well beyond the audible range for human beings. Ultrasound waves have frequencies ranging from $20 Kilohertz$ upwards.

BACKGROUND TO THE STUDY

Ultrasonic waves have various applications in today's world, especially in the medical sector. The capabilities of ultrasound in terms of medical diagnostic imaging was an outcome of the work done by two brothers, Theodore and Frederich Dussik [2]. They tried to diagnose tumors of the brain using ultrasound sometime in the 1930s and 1940s, which was around the end of the second world war.

In the late 1940s an American radiologist named Doughlas Howry [3] left his residency at Denva Veterans Administration Hospital so as to dedicate a massive portion of his time to his research in ultrasound. His intentions were to produce images of soft tissue structure using ultrasound. In 1950, Howry recorded the first cross-sectional images with utlrasound, working alongside engineers William Roderick and George Pasakonis in his garage basement.

However, these early developments were not without fault. They comprised of tanks which were used for the immersion of patients, which posed a difficulty for critically ill patients, as they couldn't be immersed in water. In 1964 the first contact two-dimensional ultrasound image was ordered, in an attempt to get an image of the cross-section of the brain. This was due to efforts a radiologist at Colombia Presbyterian Medical Center by the name of Donald King, working alongside Juan Taveras and Ray Brinker. [4].Subsequent developments in technology and techniques have pivoted Medical Ultrasonography into the advanced state it is in today.

Although technological advances played an undeniable major role in the development of ultrasound, we cannot turn a blind eye to the input that mathematics has put forth. In order to produce a high resolution picture using ultrasound, the propagation of sound must be modeled using a partial differential equation, and it is in solving such said equations that high resolution images are produced. Our focus is only on manuscripts in which finite difference method is used to solve the Westervelt equation.

The Westervelt equation is given below

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} + \frac{\beta}{\rho_0 c_0^2}\frac{\partial^2 p^2}{\partial t^2} = 0, \tag{1}$$

where p is the acoustic pressure, ρ_0 and c_0 are the ambient density and sound speed, $\beta = 1 + (B/2A)$ is the nonlinearity coefficient and (B/A) is the nonlinearity parameter.This is an empirical dimensionless parameter that is found by measurement. The first two terms in equation (1) represents the D'Alambertian operator$(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2})$ acting on the acoustic pressure (p). These describe linear lossless wave propagation at the small-signal sound speed. The final term describes the nonlinear distortion of the wave due to finite amplitude effects.

Guy V. Norton and Robert D. Purrington [6] conducted a study in which the Westervelt Equation was solved using finite difference method. In their work the Westervelt equation has higher order terms because they assumed their medium to be a thermoviscous fluid. They compared the solutions of the equation with viscous attenuaton and the equation with a convolutional/casual propagation operator. They used a fourth order accurate scheme [5] in both the space and time domain. The pressure vs time graphs for both variations agree up to a significant degree.

A thorough investigation on the formation of ultrasound image is carried out by Athananoise Karamalis and Wolfgang Wein [7]. They create a simulation that takes into account the initial pulse transmission up until the formation of the image. This is done as a way to cut costs as ultrasound machines/equipment produce pictures that depend highly on system parameters. Thus in simulating images, engineers are able to choose the optimal parameters like transducer shape. The propagation of sound is modeled using the Westervelt equation, and it is solved using the finite difference method in parallel with the Graphics processing unit in order to simulate high resolution images.

A fourth order accurate scheme is used for the space-domain, while a second-order accurarate scheme is used for the time domain. A combination of these two schemes is shown to yield impressive results.

Maxim Solovchuk and Tony W.H. Sheu [8] developed a three - dimensional acoustic thermal-hydro-dynamic coupling model so as to compute the temperature field in the hepatic cancerous region. This model considers the Westervelt equation to model the propagation of ultrasound in human tissue. The nonlinear equation is then solved using finite difference method. A sixth order accurate in three point stencil was developed to approximate the solution.

OBJECTIVES

1. Solve the Westervelt equation in regions of the domain where the nonlinearities are negligible
2. To compare the aprroximate solution with the analytical solution
3. To solve the Westervelt equation with nonlinearities using Newton-Raphson algorithm
4. Discuss the results by explaining the behaviour of the solutions as time evolves

Chapter 1

PROBLEM STATEMENT

For this work, we consider equation (1) up to the second temporal derivative, that means, we neglect the thermoviscous effects, which are significant in elecrical equipment.

PART 1

Find the numerical solution to the 1D Westervelt equation assuming the nonlinear term is negligible, that is, solve the following problem

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{1.1}$$

$$p(0,t) = p(1,t) = 0, \tag{1.2}$$

$$p(x,0) = \sin(\pi x). \tag{1.3}$$

.

PART 2

Find the numerical solution to the 1D Westervelt equation assuming the nonlinear term is non-negligible, that is, solve

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\beta}{\rho_0 c_0^2}\frac{\partial^2 p^2}{\partial t^2} = 0, \tag{1.4}$$

$$p(0,t) = p(1,t) = 0, \tag{1.5}$$

$$p(x,0) = \sin(\pi x). \tag{1.6}$$

Chapter 2

FINITE DIFFERENCE METHOD

This is a numerical approach that is often used to approximate solutions of differential equations provided the boundary conditions are known. It involves partitioning the solution domain into a finite number of divisions.

DISCRETIZING THE DOMAIN

The temporal domain $[0, T]$ is made up of a finite number of mesh points

$$0 = t_0 < t_1 < t_2 <, ..., < t_{N-1} < t_{N_t} = T, \tag{2.1}$$

whereby

$$\triangle t = t_j - t_{j-1}, 0 \leq j \leq N_t. \tag{2.2}$$

On a similar note, the spatial domain $[0, L]$ is replaced by a finite set of mesh points

$$0 = x_0 < x_1 < x_2 <, ..., < x_{N-1} < x_{N_x} = L, \tag{2.3}$$

also

$$\triangle x = x_j - x_{j-1}, 0 \leq j \leq N_x. \tag{2.4}$$

DISCRETIZING THE SOLUTION(PART 1)

Using finite difference method, we assume a solution only at the interior of the domain $[0, L] \times [0, T]$ that is

$$\frac{\partial^2 p(x_i, t_n)}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p(x_i, t_n)}{\partial t^2} = 0, \tag{2.5}$$

where $i = 1, ..., N_x - 1$ and $n = 1, ..., N_t - 1$. For $n = 0$, we have the initial condition $p(x, 0) = \sin(\pi x)$ and at the boundaries where $i = 0, N_x$ we have the boundary conditions $p(0, t) = p(1, t) = 0$.

2.0.1 DERIVATION OF THE FINITE DIFFERENCES

Suppose that p is a function of time and space such that,

$$p_i^n = p(x_i, t_n) \tag{2.6}$$

$$p_i^{n+1} = p(x_i, t_{n+1}) \tag{2.7}$$

$$p_i^{n-1} = p(x_i, t_{n-1}) \tag{2.8}$$

$$p_{i+1}^n = p(x_{i+1}, t_n) \tag{2.9}$$

$$p_{i-1}^n = p(x_{i-1}, t_n) \tag{2.10}$$

where $x_{i+1} = x_i + \triangle x$ and $t_{n+1} = t_n + \triangle t$. If we let $h = \triangle x$ and $k = \triangle t$ and expand (2.7) by Taylor expansion with the provision that $x_i = constant$, then we get

$$p_i^{n+1} = p_i^n + k\frac{\partial p}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 p}{\partial t^2} + \frac{k^3}{3!}\frac{\partial^3 p}{\partial t^3} + ... \tag{2.11}$$

Also expand (2.8) by taylor expansion provided that $x_i = constant$, then we get

$$p_i^{n-1} = p_i^n - k\frac{\partial p}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 p}{\partial t^2} - \frac{k^3}{3!}\frac{\partial^3 p}{\partial t^3} + ... \tag{2.12}$$

Now adding (2.11) and (2.12), we get that

$$p_i^{n+1} + p_i^{n-1} = 2p_i^n + k^2 \frac{\partial^2 p}{\partial t^2} + O(k^4), \tag{2.13}$$

rearranging (2.13) we arrive at the following

$$\frac{\partial^2 p}{\partial t^2} = \frac{p_i^{n-1} - 2p_i^n + p_i^{n+1}}{k^2} + O(k^2). \tag{2.14}$$

Equation (2.14) is called the central difference approximation of the second partial derivative with respect to time. Following the same procedure but this time using equations (2.9) and (2.10) we arrive at the follwing

$$\frac{\partial^2 p}{\partial x^2} = \frac{p_{i-1}^n - 2p_i^n + p_{i+1}^n}{h^2} + O(h^2). \tag{2.15}$$

Equation (2.15) is called the central difference approximation of the second derivative with respect to space. It is worth noting that the terms $O(k^2)$ and $O(h^2)$ in equations (2.14) and (2.15) respectively, imply that we have used the second order accurate central difference scheme.

DISCRETIZING THE SOLUTION(PART 2)

Just as we did in part 1, we assume a solution at the interior of the domain $[0, L] \times [0, T]$ which results into the following

$$\frac{\partial^2 p(x_i, t_n)}{\partial x^2} - \frac{1}{c_0^2} \frac{\partial^2 p(x_i, t_n)}{\partial t^2} + \frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p^2(x_i, t_n)}{\partial t^2} = 0, \tag{2.16}$$

for $i = 1, ..., N_x - 1$ and $n = 1, ..., N_t - 1$. For $n = 0$ we have the initial condition $p(x, 0) = \sin(\pi x)$ and at the boundaries where $i = 0, N_x$ we have the boundary conditions $p(0, t) = p(1, t) = 0$.

REPLACING THE DERIVATIVES BY FINITE DIFFERENCES

The second-order time derivatives are replaced by central differences.

$$\frac{\partial^2 p(x_i,t_n)}{\partial t^2} = \frac{p_i^{n-1} - 2p_i^n + p_i^{n+1}}{\triangle\ t^2}. \tag{2.17}$$

A similar approximation of the second-order derivative in the x-direction is as follows

$$\frac{\partial^2 p(x_i,t_n)}{\partial x^2} = \frac{p_{i-1}^{n} - 2p_i^n + p_{i+1}^{n}}{\triangle\ x^2}. \tag{2.18}$$

The derivation of equations (2.17) and (2.18) is oulined in [9]. Now substituting equations (2.17) and (2.18) into equation (2.5) results in the following explicit scheme for part 1

$$\lambda(p_{i-1}^n - 2p_i^n + p_{i+1}^n) - (p_i^{n-1} - 2p_i^n + p_i^{n+1}) = 0, \tag{2.19}$$

which implies

$$p_i^{n+1} = -p_i^{n-1} + 2(1-\lambda)p_i^n + \lambda(p_{i-1}^n + p_{i+1}^n), \tag{2.20}$$

where $\lambda = \frac{c_0^2 \triangle t^2}{\triangle x^2}$.

2.0.2 NEWTON-RAPHSON METHOD

Firstly we replace the derivatives in (2.6) by the central difference scheme in the same manner that we did for part 1. Thus we have the following

$$\frac{\partial^2 p(x_i,t_n)}{\partial t^2} = \frac{p_i^{n-1} - 2p_i^n + p_i^{n+1}}{\triangle\ t^2}, \tag{2.21}$$

$$\frac{\partial^2 p(x_i,t_n)}{\partial x^2} = \frac{p_{i-1}^{n} - 2p_i^n + p_{i+1}^{n}}{\triangle\ x^2}, \tag{2.22}$$

$$\frac{\partial^2 p^2(x_i,t_n)}{\partial t^2} = \frac{(p_i^{n-1})^2 - 2(p_i^n)^2 + (p_i^{n+1})^2}{\triangle\ t^2}. \tag{2.23}$$

Thus if we substitute equations (2.21),(2.22) and (2.23) into equation (2.6) and let $h =\triangle\ x$ and $k =\triangle\ t$, we get that

$$\frac{p_{i-1}^{n} - 2p_i^n + p_{i+1}^{n}}{h^2} - \frac{1}{c_0}\frac{p_i^{n-1} - 2p_i^n + p_i^{n+1}}{k^2} + \frac{\beta}{\rho_0 c_0^4}\frac{(p_i^{n-1})^2 - 2(p_i^n)^2 + (p_i^{n+1})^2}{k^2} = 0. \tag{2.24}$$

Now if we let $A = \frac{k^2 c_0^2}{h^2}$ and $B = \frac{\beta}{\rho_0 c_0^2}$ we get that

$$A(p_{i-1}^{n} - 2p_i^n + p_{i+1}^{n}) - (p_i^{n-1} - 2p_i^n + p_i^{n+1}) + B((p_i^{n-1})^2 - 2(p_i^n)^2 + (p_i^{n+1})^2) = 0. \tag{2.25}$$

Equation (2.24) is a system of nonlinear equations.Thus we suppose $\mathbf{F}$ is a function such that

$$\mathbf{F} = A(p_{i-1}^{n} - 2p_i^n + p_{i+1}^{n}) - (p_i^{n-1} - 2p_i^n + p_i^{n+1}) + B((p_i^{n-1})^2 - 2(p_i^n)^2 + (p_i^{n+1})^2). \tag{2.26}$$

This way, we can treat this as a roots finding problem where we use Newton-Raphson algorithm to approximate the different values of p such that $\mathbf{F} = 0$. In order to achieve this we neeed to compute the Jacobian matrix $\mathbf{J}$ for the system of nonlinear equations.

For this work, we consider iterations in space alone while time is held constant at the first increment. This simply says our time index, n, is held at a fixed value of 1 while the space index is allowed to run from $1 \leq i \leq N_x - 1$.

The Jacobian is defined as follows

$$\mathbf{J} = [\frac{\partial \mathbf{F}}{\partial p_i^n}] \tag{2.27}$$

where in our case $n = 1$ and $1 \leq i \leq N_x - 1$. Now we are in a position to perform the algorithm, which is given as follows

$$\mathbf{p}^{j+1} = \mathbf{p}^j - \mathbf{J}^{-1}\mathbf{F}(P^j). \tag{2.28}$$

Where $N_x = \frac{1-0}{0.001} = 1000$, this implies

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ . \\ . \\ . \\ F_{998} \\ F_{999} \end{bmatrix} =$$

$$\begin{bmatrix} Ap_0^1 + Ap_2^1 - 2(A-1)p_1^1 - p_1^0 - p_1^2 + B(p_1^0)^2 + B(p_1^2)^2 - 2B(p_1^1)^2 \\ Ap_1^1 + Ap_3^1 - 2(A-1)p_2^1 - p_2^0 - p_2^2 + B(p_2^0)^2 + B(p_2^2)^2 - 2B(p_2^1)^2 \\ Ap_2^1 + Ap_4^1 - 2(A-1)p_3^1 - p_3^0 - p_3^2 + B(p_3^0)^2 + B(p_3^2)^2 - 2B(p_3^1)^2 \\ . \\ . \\ . \\ Ap_{997}^1 + Ap_{999}^1 - 2(A-1)p_{998}^1 - p_{998}^0 - p_{998}^2 + B(p_{998}^0)^2 + B(p_{998}^2)^2 - 2B(p_{998}^1)^2 \\ Ap_{998}^1 + Ap_{1000}^1 - 2(A-1)p_{999}^1 - p_{999}^0 - p_{999}^2 + B(p_{999}^0)^2 + B(p_{999}^2)^2 - 2B(p_{999}^1)^2 \end{bmatrix}$$

and also

$$\mathbf{J} = \begin{bmatrix} -2(A-1)-4Bp_1^1 & A & . & . & . & 0 & 0 & 0 \\ A & -2(A-1)-4Bp_2^1 & A & & & & & \\ . & A & -2(A-1)-4Bp_3^1 & . & & & & \\ & . & . & . & . & & & \\ . & . & & . & . & . & & \\ & & . & . & . & . & A & \\ . & & & & & A & -2(A-1)-4Bp_{998}^1 & A \\ 0 & 0 & 0 & . & . & 0 & A & -2(A-1)-4Bp_{999}^1 \end{bmatrix}$$

2.0.3 VON NEUMANN STABILITY ANALYSIS

The stability analysis is performed to check for which values of λ do we get a stable solution and which values cause the solution to blow up. The discrete form of equation (2.5) is given by

$$p_i^{n+1} = -p_i^{n-1} + 2(1-\lambda)p_i^n + \lambda(p_{i-1}^n + p_{i+1}^n). \tag{2.29}$$

Thus the round-off error is given by

$$\in_i^n \approx (p_i^n)_{exact} - (p_i^n)_{approximate}. \tag{2.30}$$

Since (2.29) is a linear equation then

$$\in_i^{n+1} = -\in_i^{n-1} + 2(1-\lambda)\in_i^n + \lambda(\in_{i-1}^n + \in_{i+1}^n). \tag{2.31}$$

Now the Fourier representation of the round-off error is given by

$$\in_m(x,t) \approx \in_i^n = e^{at}e^{ik_m x}. \tag{2.32}$$

Stability Criterion:

$$|G| = \left|\frac{\in_i^{n+1}}{\in_i^n}\right| \leq 1. \tag{2.33}$$

substituting (2.32) into (2.31) yields;

$$e^{a(t+\triangle t)}e^{ik_m x} = -e^{a(t-\triangle t)}e^{ik_m x} + 2(1-\lambda)e^{at}e^{ik_m x} + \lambda(e^{at}e^{ik_m(x-\triangle x)} + e^{at}e^{ik_m(x+\triangle x)}) \tag{2.34}$$

dividing(2.34) by (2.32) we get that;

$$e^{a\triangle t} = -e^{-a\triangle t} + 2(1-\lambda) + \lambda(e^{-ik_m\triangle x} + e^{ik_m\triangle x}). \tag{2.35}$$

Since

$$\sin^2\left(\frac{k_m \triangle x}{2}\right) = \frac{-1}{4}(e^{-ik_m\triangle x} + e^{ik_m\triangle x} - 2). \tag{2.36}$$

This implies that

$$e^{-ik_m \triangle x} + e^{ik_m \triangle x} = 2(1 - 2\lambda \sin^2(\frac{k_m \triangle x}{2})). \tag{2.37}$$

Now

$$\frac{e^{-ik_m \triangle x} + e^{ik_m \triangle x}}{2} = (1 - 2\lambda \sin^2(\frac{k_m \triangle x}{2})). \tag{2.38}$$

Taking the absolute value of (2.38), we get that

$$-1 \leq 1 - 2\lambda \sin^2(\frac{k_m \triangle x}{2}) \leq 1, \tag{2.39}$$

which implies that

$$\lambda \sin^2(\frac{k_m \triangle x}{2}) \leq 1. \tag{2.40}$$

So finally we get that

$\lambda \leq 1.$

This implies that

$$\frac{c_0^2 \triangle t^2}{\triangle x^2} \leq 1. \tag{2.41}$$

Chapter 3

RESULTS

- In figure 3.1, we have assumed that $\triangle x = 0.001$ and $\triangle t = 0.025$. This simulation runs for a tenth of a second.
- Figure 3.2 is the analytical solution of (2.5) which is given by $p(x,t) = \cos(\pi t)\sin(\pi x)$
- Figure 3.3 is comparison of the numerical solution and the exact solution
- Figure 3.4 is the transient property of the wave,that is the wave behaviour as time evoloves.
- Figure 3.5 is the solution of the wave equation in regions of the domain where the nonlinearities are not negligible

3.0.1 DISCUSSION

In this work, we started by presenting our problem as finding the numerical solution of Westervelt equation. We then proceeded to splitting the project into two parts, in which, the first part we neglegted the nonlinear term and developed an explicit finite difference scheme to approximate the solution of the Westervelt equation. After performing the Von-Neuman stability analysis, we showed that the numerical solution agrees with the analytical solution. Lastly, for the first part, we simulated the behaviour of the wave as time evolves.

For the second part of this work, finite difference method was coupled with Newton-Raphson method to discretise and approximate the solution. The results of part 2 show a dent in the graph of the wave, this is caused by the rapid absorption of the wave energy as it propagates through human tissue. The single dent that is seen in the graph is due to the fact that we ran the algorithm for a single initial time increment.

3.0.2 CONCLUSION

The solution of the Westervelt equation was approximated in diffferent regions. Firstly a solution was found in regions where the nonlinear term was neglected. This was done using finite difference method, and the results agree with those of the analytical solution. The behaviour of the wave was simulated for different time steps. Secondly, finite difference method together with Newton-Raphson method were used to approximate the solution of the Westervelt equation including the nonlinear term. By considering the three dimensional form of the Westervelt equation and higher order finite difference approximation schemes coupled with graphics technology we can open the door to high quality resolution pictures in Medical ultrasound.

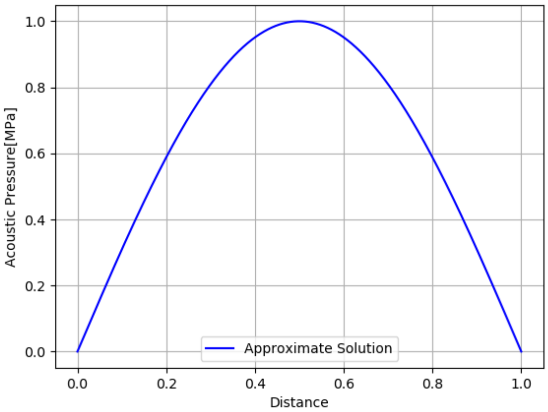

Figure 3.1: NUMERICAL SOLUTION

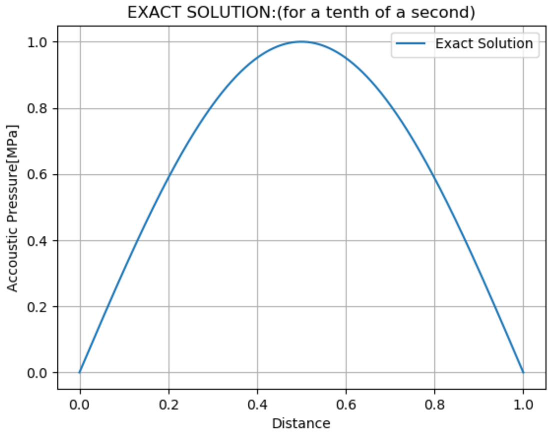

Figure 3.2: EXACT SOLUTION

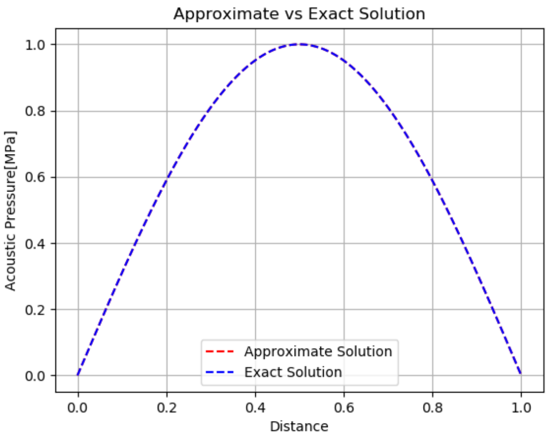

Figure 3.3: NUMERICAL vs EXACT

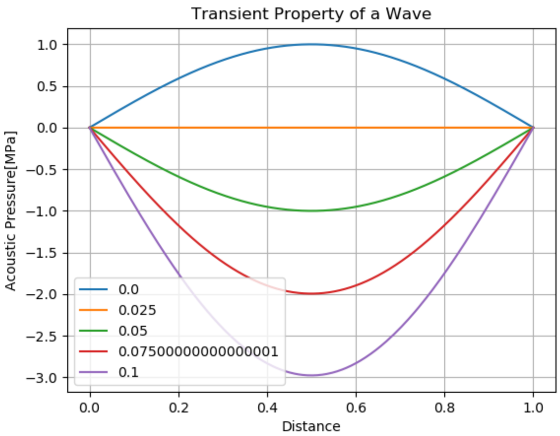

Figure 3.4: TRANSIENT SOLUTION

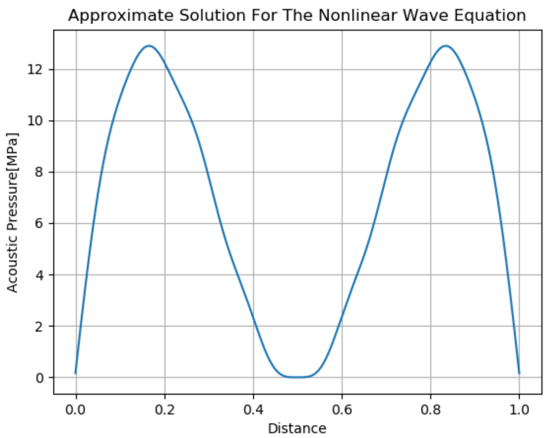

Figure 3.5: NONLINEAR SOLUTION

Appendix A

PYTHON CODES FOR THE GRAPHICAL SOLUTIONS

THE CODE BELOW IS FOR THE FIRST IMAGE

```
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1
c = 0
d = 0.1
c_not = 1


h = 0.0001
k = 0.025



x = np.arange(a,b+h,h)
t = np.arange(c,d+k,k)
boundaryconditions = [0, 0]
initialconditions = np.sin(np.pi*x)

n = len(x)
m = len(t)
factor = (c_not**2*k**2)/h**2
```

```
P = np.zeros((n,m))
P[0,:] = boundaryconditions[0]
P[-1,:] = boundaryconditions[1]
P[:,0] = initialconditions


for j in range (1,m-1):
for i in range(1,n-1):
P[i,j+1] = -P[i,j-1] + 2*(1 - factor)*P[i,j] + factor*(P[i-1,j] + P[i+1,j])



plt.plot(x,P[:,0],'-b',label = 'Approximate Solution')

plt.xlabel('Distance')
plt.ylabel('Acoustic Pressure[MPa]')
plt.legend()
plt.grid(True)
plt.show()
```

THE CODE BELOW IS FOR THE SECOND IMAGE

```
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1
c = 0
d = 0.1
c_not = 1


h = 0.0001
k = 0.025



x = np.arange(a,b+h,h)
t = np.arange(c,d+k,k)

n = len(x)
m = len(t)
```

```
boundaryconditions = [0, 0]
initialconditions = np.sin(np.pi*x)

u = np.zeros((n,m))

for j in range(m):
for i in range(n):
u[i,j] = np.cos(np.pi*t[j])*np.sin(np.pi*x[i])

plt.plot(x,u[:,0],'--b', label = 'Exact Solution')
plt.xlabel('Distance')
plt.ylabel('Acoustic Pressure[MPa]')
plt.legend()
plt.grid(True)
plt.show()
```

THE CODE BELOW IS FOR THE THIRD IMAGE

```
import numpy as np
import matplotlib.pyplot as plt



a = 0
b = 1
c = 0
d = 0.1
c_not = 1


h = 0.0001
k = 0.025



x = np.arange(a,b+h,h)
t = np.arange(c,d+k,k)
boundaryconditions = [0, 0]
initialconditions = np.sin(np.pi*x)

n = len(x)
m = len(t)
factor = (c_not**2*k**2)/h**2
```

```
u = np.zeros((n,m))

for j in range(m):
        for i in range(n):
        u[i,j] = np.cos(np.pi*t[j])*np.sin(np.pi*x[i])


P = np.zeros((n,m))
P[0,:] = boundaryconditions[0]
P[-1,:] = boundaryconditions[1]
P[:,0] = initialconditions


for j in range (1,m-1):
for i in range(1,n-1):
P[i,j+1] = -P[i,j-1] + 2*(1 - factor)*P[i,j] + factor*(P[i-1,j] + P[i+1,j])



plt.plot(x,P[:,0],'--r',label = 'Approximate Solution')
plt.plot(x,u[:,0],'--b', label = 'Exact Solution')
plt.title('Approximate vs Exact Solution')
plt.xlabel('Distance')
plt.ylabel('Acoustic Pressure[MPa]')
plt.legend()
plt.grid(True)
plt.show()
```

THE BELOW IS FOR THE FOURTH IMAGE

```
import numpy as np
import matplotlib.pyplot as plt


a = 0
b = 1
c = 0
d = 0.1
c_not = 1


h = 0.0001
k = 0.025
```

```
x = np.arange(a,b+h,h)
t = np.arange(c,d+k,k)
boundaryconditions = [0, 0]
initialconditions = np.sin(np.pi*x)

n = len(x)
m = len(t)
factor = (c_not**2*k**2)/h**2


P = np.zeros((n,m))
P[0,:] = boundaryconditions[0]
P[-1,:] = boundaryconditions[1]
P[:,0] = initialconditions


for j in range (1,m-1):
for i in range(1,n-1):
P[i,j+1] = -P[i,j-1] + 2*(1 - factor)*P[i,j] + factor*(P[i-1,j] + P[i+1,j])



plt.plot(x,P)
plt.title('Transient Property of a Wave')
plt.xlabel('Distance')
plt.ylabel('Acoustic Pressure[MPa]')
plt.legend(t)
plt.grid(True)
plt.show()
```

THE CODE BELOW IS FOR THE LAST IMAGE

```
import numpy as np
import matplotlib.pyplot as plt

#boundary parameters
a = 0
b = 1
c = 0
d = 0.1

#stepsizes
```

```
h = 0.001
k = 0.025

x = np.arange(a,b+h,h)
t = np.arange(c,d+k,k)
boundaryconditions = [0, 0]
initialcondions = np.sin(np.pi*x)

n = len(x)
m = len(t)


#solution meshgrid with boundary conditions
p = np.zeros((n,m))
p[0,:] = boundaryconditions [0]
p[-1,:] = boundaryconditions [1]
p[:,0] = initialcondions
factor = k**2/h**2
beta = 1


#function of non-linear equations
F = np.zeros((n))


def Func(p,factor,beta):

for i in range (1,n-1):

F[i] = factor*p[i-1,1] + factor*p[i+1,1] - 2*(factor - 1)*p[i,1] - p[i,0] -
 p[i,2] + beta*p[i,0]**2 + beta*p[i,2]**2 - 2*beta*p[i,1]**2

return F


#Jacobian Matrix
A = np.zeros((n,n))
A[0,0] = -2*(factor - 1) - 4*beta*p[1,1]
A[-1,-1] = -2*(factor - 1) - 4*beta*p[n-1,1]
A[0,1] = factor
A[1,0] = factor

for j in range(1,n-1):

A[j,j] = -2*(factor - 1) - 4*beta*p[j,1]
```

```
A[j+1,j] = factor
A[j,j+1] = factor


#colounm matrix of pressure values
def p_value(p):

P = np.zeros((n))

for i in range (n):

P[i] = p[i,1]

return P



#Newton-Raphson Algorithm
def Newton(Func,factor,beta,A,p,eps):

prs = p_value(p)
F_value = Func(p,factor,beta)
F_norm = np.linalg.norm(F_value, ord = 2)
iteration_counter = 0


while abs(F_norm) > eps and iteration_counter < 100:
delta = np.linalg.solve(A,-F_value)
prs = prs + delta
F_value = Func(p,factor,beta)
F_norm = np.linalg.norm(F_value,ord = 2)
iteration_counter += 1

if abs(F_norm) > eps:
iteration_counter = -1
return prs


solution = Newton(Func,factor,beta,A,p,eps = 0.0001)

plt.plot(x,solution)
plt.xlabel('Distance')
plt.ylabel('Acoustic Pressure[MPa]')
plt.title('Approximate Solution For The Nonlinear Wave Equation')
plt.grid(True)
```

```
plt.show()
```

Bibliography

[1] O. Rudenko, "Nonlinear acoustics in medicine: A review," *Physics of Wave Phenomena*, vol. 30, no. 2, pp. 73–85, 2022.

[2] K. A. Kaproth-Joslin, R. Nicola, and V. S. Dogra, "The history of us: from bats and boats to the bedside and beyond: Rsna centennial article," *Radiographics*, vol. 35, no. 3, pp. 960–970, 2015.

[3] C. F. Dietrich, L. Bolondi, F. Duck, D. H. Evans, C. Ewertsen, A. G. Fraser, O. H. Gilja, C. Jenssen, E. Merz, C. Nolsoe, *et al.*, "History of ultrasound in medicine from its birth to date (2022), on occasion of the 50 years anniversary of efsumb. a publication of the european federation of societies for ultrasound in medicine and biology (efsumb), designed to record the historical development of medical ultrasound," *Medical Ultrasonography*, vol. 24, no. 4, pp. 434–450, 2022.

[4] B. B. Goldberg, R. Gramiak, and A. K. Freimanis, "Early history of diagnostic ultrasound: the role of american radiologists," *AJR Am J Roentgenol*, vol. 160, no. 1, pp. 189–194, 1993.

[5] A. Bayliss, K. Jordan, B. LeMesurier, and E. Turkel, "A fourth-order accurate finite-difference scheme for the computation of elastic waves," *Bulletin of the Seismological Society of America*, vol. 76, no. 4, pp. 1115–1132, 1986.

[6] R. D. Purrington and G. V. Norton, "A numerical comparison of the westervelt equation with viscous attenuation and a causal propagation operator," *Mathematics and Computers in Simulation*, vol. 82, no. 7, pp. 1287–1297, 2012.

[7] A. Karamalis, W. Wein, and N. Navab, "Fast ultrasound image simulation using the westervelt equation," in *Medical Image Computing and Computer-Assisted Intervention–MICCAI 2010: 13th International Conference, Beijing, China, September 20-24, 2010, Proceedings, Part I 13*, pp. 243–250, Springer, 2010.

[8] M. Solovchuk, T. W. Sheu, and M. Thiriet, "Simulation of nonlinear westervelt equation for the investigation of acoustic streaming and nonlinear propagation effects," *The Journal of the Acoustical Society of America*, vol. 134, no. 5, pp. 3931–3942, 2013.

[9] N. J. Nchejane, "Soliton-like humps on the surface of magnetic fluids," 2008.