Application of Lie Symmetries to solving fractional Black-Scholes option pricing model in financial mathematics

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DECLARATION

I, Realeboha Ramoeletsi, declare that this mini dissertation entitled, "Application of Lie Symmetries to solving fractional Black-Scholes option pricing model in financial mathematics" represents my own work which has been done for degree of Masters in applied mathematics. I have read the university, faculty, and department regulations, and accept the responsibility for the conduct of the procedures in accordance with university's regulations, and where I have consulted the published work, this has been clearly attributed observing the university policy on plagiarism.

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ABSTRACT

We perform Lie symmetry analysis to the fractional Black-Scholes option pricing model whose price evolution is described in terms of a partial differential equation (PDE). As a result, new complete Lie symmetry group and infinitesimal generators of the one-dimensional fractional Black-Scholes pricing model are derived. Furthermore, we compute a family of exact invariant solutions that constitute the pricing models for the Black-Scholes model using the associated infinitesimal generators and the corresponding similarity reduction equations. Using known solutions, more solutions are generated via group point transformations.

KEYWORDS

Lie point symmetries; Fractional Black-Scholes; Option pricing; Financial mathematics.

DEDICATION

I devote my paper work to my family and numerous companions. A extraordinary feeling of appreciation to my cherishing guardians, Mabokang Celina Ramoeletsi and Samuel Paepae Ramoeletsi whose words of support and thrust for relentlessness ring in my ears. My sister Moliehi, My brothers Bokang and Reanetse have never cleared out my side and are exceptionally uncommon.

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INDEX OF ABBREVIATIONS

LHS: Left hand side.

RHS: Right hand side.

Iff: if and only if.

etc: Et cetera.

PDEs: Partial differential equations.

ODEs: Ordinary differential equations.

FPDE: Fractional partial differential equation.

FDM: Finite difference method.

FEM: Finite element method.

ADM: Adomain decomposition method.

HPM: Homotopy perturbation method.

HAP: Homotopy analysis method.

FDTM: Fractional differential transformation method.

MFDTM: Modified fractional differential transformation method.

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Chapter 1

Introduction and background

For centuries, financial option world was facing uncertainty and risks which were impossible to analyse, until Black-Scholes (1973) came up with the Black-Scholes partial differential equation (Black-Scholes PDE). This equation was used as a model of option pricing and it is now famously considered in history of modern finance according to McKay [35]. The name Black-Scholes came by an economist Robert Merton. However, the nature of Black-Scholes equation

$$\frac{\partial u}{\partial t} + \frac{x^2\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0, \tag{1.1}$$

is named after two scientists: Fisher Black and Myron Scholes. According to Henderson [15], these three scientists (Fisher Black, Myron Scholes and Robert Merton) worked together for years leading to the publication of Black Scholes model in 1973. Years after the publication, Fisher Black, Myron Scholes and Robert Merton were awarded a Nobel Prize for their excellent work. The extension of Black-Scholes was then handled by Mark German and Steven Kohhagen in foreign currency option values [8]. After the discovery and extension of this remarkable work, other researchers began investigating the existence of solutions of Black-Scholes model(1.1) by implementing different methods [1–3, 5, 10].

Fractional calculus began to show its significance by giving attention to various science and engineering disciplines [11, 38]. As a result, many researchers fully participated and contributed in this field. Moreover, the book by Mir Saijjad Hashemi and Dumitru Baleanu [12] has played an important part such as group analysis and exact solutions of fractional partial differential equations in formulation of the subject. Considering the use of known fractional derivatives such as: Grunwald-Letnikow, Riemann-Liouville $\frac{\partial^\alpha u}{\partial t^\alpha}$, Hadamard, Caputo and Riesz, interesting results such as group analysis of time-fractional Fokker-Planck, Lie symmetry of time-fractional Fisher equation, Lie symmetry of time-fractional $K(m,n)$ equation, Lie symmetry of time-fractional gas dynamic equation, etc in relation to fractional differential equations can be found in this book and references therein.

Nowegian mathematician Marius Sophus Lie introduced Lie group analysis solutions involving ODEs and PDEs [32, 36]. He considered the use of variable transformations leading to reduction of a differential equation in invariant form. Combining fractional calculus, Black-Scholes and Lie symmetries, researchers such as [21, 22, 39, 43] were able to identify a relationship between these methods and that led to implementation of mathematical techniques for analysis involving these methods. Wyss et al [43] used fractional Black-Scholes equation considering fractional derivatives to price European call option. Not only Wyss et al considered the use of fractional Black-Scholes equation considering fractional derivatives to price European call option, but also, Lina Song and Weiguo Wang [39] used Lie symmetries with finite difference method technique to analyse solution of the fractional Black-Scholes option pricing Model. In their work, Lina Song and Weiguo Wang [39] considered option price $u = u(x,t)$ subject to the time fractional Black-Scholes in the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = (ru - rx\frac{\partial u}{\partial x})\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} \qquad t>0, 0<\alpha \leq 1, \tag{1.2}$$

where $r,\ \alpha,\ \Gamma$ denote the risk free interest rate, volatility and Gamma function, respectively, and $\frac{\partial^\alpha u}{\partial t^\alpha}$ denotes the Riemann-Lioville [12].

Finite difference method as one of the methods used to solve PDEs was put in to establish the importance of Lie symmetry in finance by implementing Lie symmetries to analyse the fractional Black-Scholes option model. Lina Song and Wieguo Wang [39] modified this work by combining Jumaries time-fractional Black-Scholes equation, the terminal and boundary conditions satisfied by the standard put price to investigate the pricing problems for European and American put options based on a fractional derivative called modified Riemann-Lioville fractional derivative $\frac{\partial^{\alpha} u}{\partial t^{\alpha}}$. In addition, Lina Song and Wieguo Wang [39] in their work stated that, to get rid of arbitrage, the option value at each point in $u(x,t)$ space should be greater than the intrinsic value.

Recently, group invariant especially the Lie points symmetry have been put in financial market [4,9,31]. To ascertain the use of Lie symmetry, however, fractional derivatives are of more interest because of the computational techiques and procedures used in solving Black-Scholes (1.1). Unlike classical derivatives, fractional derivatives use Riemann-Liouville definition for computation. Huang et al [17] implemented Lie symmetry and exact solutions considering the time fractional Harry-Dym equation with Riemann–Liouville derivative $\frac{\partial^{\alpha} u}{\partial t^{\alpha}}$. Application of Lie symmetries is considered as an alternative way to solving PDEs. There are many methods used to solve PDEs such as Finite Difference Method (FDM), Finite Element Method (FEM), Adomain Decomposition Method (ADM) and methods of lines to name the few. According to Qiu et al [33], non linear PDEs are more challenging but interesting in this discipline. Due to sensational scope and applications in several studies, this has alarmed the need to conduct research in attempt to analyze solutions especially in fractional Black-Scholes.

Moreover, Chong et al [7] used fractional differential operator Riemann-Lioville derivative $\frac{\partial^{\alpha} u}{\partial t^{\alpha}}$ to analyse Lie symmetries of fractional Black-Scholes equation. The main aim of Chong et al's work was to implement hedging strategy to minimize the risks in financial market, constructing infinitesimal operators and obtaining results which were easy to interpret and analyse. Invariant conditions were constructed then obtained determining equations which were easily integrated to transform variable remaining to Riemann-Lioville fractional derivative operator $\frac{\partial^{\alpha} u}{\partial t^{\alpha}}$ structure thus, spanning fractional Black-Scoles by vector fields [7]. Lie point symmetries were also obtained but, due to compound and complicated infinitesimal operators, only the first infinitesimal generator, X_1 was considered. Chong also pointed out in his work that, the solution obtained was trivial on financial perspectives, so the results were compared and interpreted.

Fascinating problems such as combination of Laplace perturbation method and homotopy perturbation by Kumar et al [28], however, are observed in solving non linear PDEs in financial mathematics [33]. The effectiveness of this combination led to, not only accurate, but also precise interpretable results to fractional Black-Scholes equation for European option pricing problem with boundary conditions. Reduction of compound and complicated numerical computations came as a result of incurred round off errors. For further analysis, they employed two examples in their method and used He's polynomials as well as converging power series considering Black-Scholes equation (1.1) in two non linear fractional differential equations to elucidate how effective their method was. To summarize, Laplace perturbation method and homotopy perturbation bridles the deficiency generated by unsatisfied conditions hence why it is very powerful and efficient in approximation and computing numerical solutions in finance. Not only Jagdev et al [28] considered this strategy of solving fractional Black-Scholes but also Kumar et al [27] used Homotopy perturbation method (HPM) introduced by HE [13, 14]. Moreover, Homotopy analysis method (HAP) implemented by Liao and Shi-Jun [29, 30] was also used to analyse numerical solution for time fractional Black-Scholes with boundary conditions for European option problem. This method gave well displayed results on HAP in comparison with HPM.

Kanth et al [24] presented FDTM (Fractional differential transform method) together with MFDTM (Modified fractional differential transform method) for analysis of fractional Black-Scholes European option pricing equation. These methods do not require restrictions, transformation and discretization as compared to other methods discussed such as FDM, ADM, HPM and VIM. ADM is complex since it requires computation of adomain polynomials and errors are likely to occur during computation. The difficulty in VIM has an inherent inaccuracy on identifying the Lagrange multiplier, correctional functional and stationary conditions for the fractional order. Also, a disadvantage of HPM is solving functional equation in each iteration, which is sometimes complicated and unattainable [27]. Therefore, the proposed FDTM and MFDTM are much easier when compared to ADM, VIM and HPM. The use of FDTM and MFDTM turned to be much effective in consideration of boundary conditions and this effectiveness is summarized with the use of examples [24]. Since FDTM deals mainly with Taylor series expansion for all variables, it encounters problems while considering non linear functions. For this reason, MFDTM is used to reduce complications in solving non linear differential equations of fractional order in finance [27]. The issue of risks and uncertainty in financial industry is not only uncontrollable but also compound and impossible to analyse. Option in finance play an important role when it comes to good trading of assets, investment and risks management. As a result, this issue has alarmed the need to conduct researches in attempt to analyse the solution in fractional Black-Scholes using different mathematical tools and methods. Upon the use of equation (1.2), the aim of this work is to extent and implement Lie symmetry technique to solve fractional Black-Scholes option pricing model in financial mathematics. Following the work by Song et al [39], we consider the use of option price u subject to the time fractional Black-Scholes equation. Using this form of equation, determining systems of equations are obtained, and from this system, resulting infinitesimals are constructed. Making use of obtained determining equations, we compute the invariant solutions of our family equation.

1.1 Aim of the study

In this work, we aim to price fractional Black-Scholes model by means of solving associated fractional partial differential equation (FPDE) by making use of Lie symmetry analysis.

1.2 Objectives of the study

The objectives of the study are as follows:

- We first find the determining systems of equations of equation (1.2).
- Solve each system to find the infinitesimals.
- Use the resulting infinitesimals to find the invariant solution.
- Finally, graph the results and interpret them in relation to classical Black-Scholes equation(1.1).

1.3 The work flow of the current study.

We start by stating useful functions and their basic properties in relation to fractional derivatives and Lie symmetry in chapter 2. We continue by solving our family equation (1.2) in chapter 3 using information from chapter 2 and then find the invariant solutions of equation (1.2). In chapter 4 we give graphical representations of our solutions and interpretations of the results obtained. We finally give the discussion and conclusion in chapter 5 which is followed by MATLAB, Maple codes and proofs useful in this work found in section 5.

Chapter 2

Mathematical Preliminaries

In this chapter, we give some definitions and basic properties that have been employed throughout our dissertation. We define notions such as Riemann-Liouville $\frac{\partial^\alpha u}{\partial t^\alpha}$, Gamma function, Black-Scholes, Lie symmetry, Lie brackets as well as Option pricing .

2.1 Definitions and Basic properties

Fractional calculus

2.1.1 Riemann-Liouville

Definition 1. Riemann-Liouville [12, 38, 44] of order α is defined by:

$$D_t^\alpha u(t,x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{u(\xi,x)}{(t-\xi)^{\alpha+1-m}} \, d\xi & 0<m-1<\alpha \leq m, m \in N \\ \frac{\partial^n u}{\partial t^n} & \alpha = n \end{cases}$$

where $\frac{\partial^\alpha u}{\partial t^\alpha}$ denotes the α^{th} fractional derivative and Γ denotes a Gamma function.

2.1.2 Gamma function

A well known mathematician L. Euler (1729) came up with a fascinating function know as Gamma function represented symbolically by $\Gamma(.)$. This function is used cooperatively with factorial (!) of all positive real numbers $\mathbb{R}^+$.

Gamma function

Upon the use of gnuplot 40 and MATLAB, the gamma fuction is shown on the figure below:

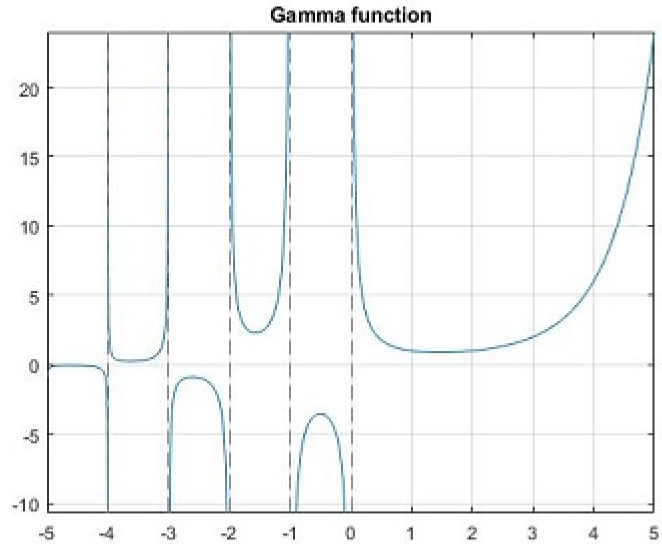

Figure 2.1: Gamma function along real axis

2.1.3 The Mittag-Leffler Function

Introduction

A Swedish mathematician [25] came up with an interesting function known as a Mittag-Leffler funtion defined in terms of a power series as

$$E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)} \quad \alpha>0. \tag{2.1}$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \quad \alpha>0, \quad \beta>0. \tag{2.2}$$

Mittag-Leffler funtion is defined directly from the exponential function e^x, which was futher modified in 1950's by R.P Agerwal [34].

Given $x,\ \alpha,\ \beta \in \mathbb{R}$, the following properties hold:

Properties

$E_{\alpha,\beta}(0) = 1,$

$$
\begin{aligned}
E_{1,1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} \\
&= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\
&= e^x, \\
E_{1,2}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} \\
&= \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} \\
&= \frac{e^{x-1}}{x}, \\
E_{\frac{1}{2},1}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\frac{k}{2}+1)} \\
&= e^{x^2} Erfc(-x) \ \text{ where } Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt \ \ x \in \mathbb{R}.
\end{aligned}
$$

2.1.4 Laplace transformation of fractional derivatives

The Laplace transformation of y^n is given by [25]

$$
\begin{aligned}
\mathscr{L}\{y^n\} &= s^n Y - s^{n-1} y(0) - s^{n-2} y'(0) - ... - y^{n-1}(0) \\
&= s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^k(0). \qquad (2.3)
\end{aligned}
$$

We now define a fractional derivative of $y(t)$ as

$$D^\alpha y(t) = D^n(D^{-\mu} y(t)). \qquad (2.4)$$

By letting n be the smallest integer greater than $\alpha > 0$ and $\mu = n - \alpha$, (2.4) becomes

$$D^\alpha y(t) = D^n(D^{-(n-\alpha)} y(t)). \qquad (2.5)$$

Suppose that $y(t)$ exists, now, taking the Laplace transform of (2.5) yields

$$
\begin{aligned}
\mathscr{L}\{D^\alpha y(t)\} &= \mathscr{L}\{D^n(D^{-(n-\alpha)} y(t))\} \\
&= s^n \mathscr{L}\{D^{-(n-\alpha)} y(t)\} - \sum_{k=0}^{n-1} s^{n-k-1} D^k(D^{-(n-\alpha)} y(t))|_{t=0}
\end{aligned}
$$

$$= s^n[s^{-(n-\alpha)}Y(s)] - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-n+\alpha} y(0)$$

$$= s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} D^{k-n+\alpha} y(0). \tag{2.6}$$

For $n = 1, 2$, (2.6) reduces to (2.7) and (2.8) respectively

$$\mathscr{L}\{D^{\alpha}y(t)\} = s^{\alpha}Y(s) - D^{-(1-\alpha)}y(0) \ \text{ for } \ 0<\alpha \leq 1. \tag{2.7}$$

$$\mathscr{L}\{D^{\alpha}y(t)\} = s^{\alpha}Y(s) - sD^{-(2-\alpha)}y(0) - D^{-(1-\alpha)}y(0) \ \text{ for } \ 1<\alpha \leq 2. \tag{2.8}$$

In this work however, we are interested in the value of α such that $0<\alpha \leq 1$.

Below is the summary table of Laplace transform parts useful in this work, where $b \neq a$ are real constants and $\alpha, \ \beta>0$.

Laplace transform pairs

$y(t)$	$y(t) = \mathscr{L}^{-}Y(S)$
$\frac{1}{s^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{(s-a)(s-b)}$	$\frac{e^{at}-e^{bt}}{a-b}$
$\frac{1}{s^{\alpha}-a}$	$t^{\alpha-1}E_{\alpha,\alpha}(at^{\alpha})$
$\frac{1}{s^{\alpha}(s-a)}$	$t^{\alpha}E_{1,\alpha+1}(at)$
$\frac{a}{s(s^{\alpha}+a)}$	$1 - E_{\alpha}(-at^{\alpha})$
$\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$	$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$
$\frac{1}{(s+a)^{\alpha}}$	$\frac{t^{\alpha-1}e^{-at}}{\Gamma(\alpha)}$
$\frac{s^{\alpha}}{s(s^{\alpha}+a)}$	$E_{\alpha}(-at^{\alpha})$
$\frac{s^{\alpha-\beta}}{(s-a)^{\alpha}}$	$\frac{t^{\beta-1}}{\Gamma(\beta)}F_1(\alpha; \ \beta; \ at)$

2.1.5 Basic definitions in relation to Lie symmetry analysis of a fractional differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F(x,t,u,u_x,u_{xx},...) \qquad 0<\alpha \leq 1. \tag{2.9}$$

One-parameter Lie group of transformations are represented by:

$$\bar{t} = t + \epsilon\xi^1(x,t,u) + \mathcal{O}(\epsilon^2),$$

$$\bar{x} = x + \epsilon\xi^2(x,t,u) + \mathcal{O}(\epsilon^2),$$

$$\bar{u} = u + \epsilon\phi(x,t,u) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial^\alpha \bar{u}}{\partial t^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\phi^0_\alpha(t,x,u) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial u}{\partial x} + \epsilon\phi^x(t,x,u) + \mathcal{O}(\epsilon^2),$$

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\partial^\alpha u}{\partial x^2} + \epsilon\phi^{xx}(t,x,u) + \mathcal{O}(\epsilon^2),$$

where ϕ^x and ϕ^{xx} are defined as:

$$\phi^x = D_x(\phi) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \tag{2.10}$$

$$\phi^{xx} = D_x(\phi^x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \tag{2.11}$$

$$\phi^{xxx} = D_x(\phi^x x) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2), \tag{2.12}$$

and the total derivatives are:

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_t} + u_{xt}\frac{\partial}{\partial u_x} + ...$$

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{tx}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + ...$$

The infinitesimal generator X of equation (2.9) can be expressed as

$$X = \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial x} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{2.13}$$

where ξ^1, ξ^2 and ϕ are infinitesimals to be determined.

X is said to be a symmetric generator of (2.9) provided the prolongation

$$Pr^{\alpha,2}X(\triangle)|_{\triangle=0} = 0 \qquad \text{where } \triangle = \frac{\partial^\alpha u}{\partial t^\alpha} - F(x,t,u,u_x,u_{xx}). \tag{2.14}$$

The α^{th} Prolongation operator $Pr^{\alpha,2}$ takes the form

$$\begin{aligned} Pr^{\alpha,2} &= X + \phi_\alpha^0 \frac{\partial}{\partial t^\alpha u} + \phi^x \frac{\partial}{\partial v_x} + \phi^{xx} \frac{\partial}{\partial u_{xx}} \\ &= \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial s} + \phi(x,t,u)\frac{\partial}{\partial t} + \phi_\alpha^0 \frac{\partial}{\partial t^\alpha u} + \phi^x \frac{\partial}{\partial u_x} + \phi^{xx}\frac{\partial}{\partial u_{xx}}. \end{aligned} \tag{2.15}$$

Also α^{th} infinitesimal has the form

$$\phi_\alpha^0 = D_t^\alpha + \xi^2 D_t^\alpha(u_x) - D_t^\alpha(\xi^2 u_x) + D_t^\alpha(D_t(\xi^1)u) - D_t^{\alpha+1}(\xi^1 u) + \xi^1 D_t^{\alpha+1}(u), \tag{2.16}$$

where D_t^α is the total fractional derivative operator.
By Leibniz rule [12], we have that (2.16) reduces to

$$\begin{aligned} \phi_\alpha^0 = \frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\xi^1))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi_u}{\partial t^\alpha} + \sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{a}{n+1} D_t^{n+1}(\xi^1)\right] D_t^{\alpha-n}(u) - \\ \sum_{n=1}^{\infty}\binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n}(u_x) + \mu] \end{aligned} \tag{2.17}$$

,where,

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{a}{n}\binom{n}{n}\binom{k}{r}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}(-u)^r\frac{\partial^m}{\partial t^m}(u^{k-r})\frac{\partial^{n-m+k}}{\partial t^{n-m}\partial u^k},$$

note that $\mu = 0$ since $\frac{\partial^k \phi}{\partial u^k} = 0$ for $k \geqslant 2$, here the infinitesimal ϕ is linear in the variable u and μ.

Some useful properties of fractional derivative operator

For any $0<\alpha \leq 1$ and functions $f,\ g$ being the α^{th} differentialble functions at a point $t>0$ [**?**], the following are equivalent:

1. $D^\alpha[af(t) + bg(t)] = aD^\alpha[f(t)] + bD^\alpha[g(t)] \qquad \forall\ a, b\ \in \mathbb{R}.$
2. $D^\alpha[f(t)g(t)]=f(t)D^\alpha[g(t)] + g(t)D^\alpha[f(t)].$
3. $D^\alpha[\frac{f(t)}{g(t)}]=\frac{g(t)D^\alpha[f(t)]-f(t)D^\alpha[g(t)]}{[g(t)]^2}.$
4. For $f(t)=t^x$, $D^\alpha[f(t)]=xt^{x-\alpha}$, $\forall x\ \in \mathbb{R}.$
5. $D^\alpha[c]=0$, for c being a constant.
6. $D^\alpha y(t)=D^n(D^{-\mu}y(t))$

2.1.6 Group invariant solution

Just like partial differential equations (PDEs), the invariant solution of a frational differential equation is defined by considering a function $u = u(x,t)$ which is said to be an invariant solution under fractional differential equation in relation to the infinitesimal operator (2.13) iff

$$\xi^1(u,x,t)u_t + \xi^2(u,x,t)u_x = \phi(u,t,x).$$

With assumption that ξ^1, ξ^2 not being zeros, we make use of a characteristic method:

$$\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du}{\phi}. \tag{2.18}$$

Letting two arbitrary differentiable funtions $p(u,x,t)$ and $q(u,x,t)$ with $q_u \neq 0$ be independent first integral functions of (2.18), we turn to have a general solution of the invariant condition as $q = F(p)$. We then solve for F by substituting this solution into our original FDE (1.2). One should note that the resulting equation after substitution may either be solvable or not solvable depending on the nature of an equation obtained.

Definition 2. A function $u = \theta(x,t)$ is said to be an invariant solution of (2.9) with respect to the infinitesimal (2.13) iff the following hold:

1. $u = \theta(x,t)$ satisfies (2.9),
2. $u = \theta(x,t)$ is a solution of (2.9).

2.1.7 Lie algebra

Definition 3. [36] Given a vector space over a field F, we define a Lie algebra as $\Big[,\Big]$ and it is denoted by $\mathfrak{L}$. Given a bilinear commutation law, the following properties have to be satisfied:

1. *closure*: For X,Y $\in \mathfrak{L}$, we have that $\Big[X,Y\Big] \in \mathfrak{L}$.
2. *Bilinearity*: $\Big[X, \alpha Y + \beta Z\Big] = \alpha\Big[X,Y\Big] + \beta\Big[X,Z\Big]$, where $\alpha, \beta \in\ F$ and $X, Y, Z \in \mathfrak{L}$.
3. *Jacobi identity:* $\Big[X,\Big[Y,Z\Big]\Big] + \Big[Y,\Big[Z,X\Big]\Big] + \Big[Z,\Big[X,Y\Big]\Big] = 0.$
4. *Skew-symmetry:* $\Big[X,Y\Big] = -\Big[Y,X\Big].$

2.1.8 Introduction on Black-Scholes and fractional Black-Scholes equation

In this section, we give brief definations of Black-Scholes equation(1.1) and fractional Black-Scholes equation(1.2) and how they relate. All the information in this section is extracted from Jiang and Lishang [20].

Black-Scholes equation(1.1)

Black-Scholes equation is defined as a second order partial differential equation. Black-Scholes equation is said to be of much importance in financial mathematics since it minimizes risks such as hedging.

Definition 4. The general European call or put options on an underlying stock paying no dividends. Black-Scholes PDE is stated as follows

$$\frac{\partial u}{\partial t} + \frac{x^2\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0,$$

where $u(s,t)$ is European option price, r is risk free rate, x is the stock price and σ is volatility of the stock.

Assumptions of Black-Scholes by Hull [18].

1. Securities can be sold short with full use of proceeds.
2. There are no transaction costs or taxes. All securities are perfectly divisible.
3. There are no dividends paid during the life of the contract.
4. Risk-less arbitrage opportunities are not permissible.
5. Securities are traded continuously.
6. The risk-free interest rate, r, is constant throughout the life of a security.

The Black-Scholes equation is derived using Ito's formula, the derivation of Black-Scholes equation(1.1) is clearly shown step-by-step in [20]. Howevere, our main focus in this work, is the fractional order of Black-Scholes equation. We consider the fractional part of the derivatives, that is, non integer order say α. One can clearly see that, when $\alpha = 1$ in equation (1.2), we have a Black-Scholes PDE. This is how classical Black-Scholes and fractional Black-Scholes(1.1) relate.

fractional Black-Scholes equation

Definition 5. In general, fractional Black-Scholes equation of order α is defined as;

$$\frac{\partial^\alpha u}{\partial t^\alpha} + \frac{x^2\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0,$$

Just like Black-Scholes equation PDE(1.1), there are several approaches used to derive the fractional Black-Scholes PDE such as, equation of evolution approach by Wyss et al [42], fractional Taylor series method by Kumar [28] and Laplace Homotopy Perturbation approach by Jumarie et al [23]. Employing definitions, 4, 5 and making use of equation (1.2) by [39], in this work, we solve equation (1.2) by means of Lie symmetry. Our main focus is also on European option pricing.

2.1.9 Financial derivative

A financial derivative or a contingent claim is defined as a contract whose value is determined by the value of an underlying asset. There are several tools used in financial derivatives, to name the few, we have risk, arbitrage, volitality, etc Obeng [37].

Financial derivative tools [18]

Risk: is defined in portfolio as a variance of the return.

Arbitrage: is defined in finance as a way of making surplus without taking risks.

Volitality: σ is assumed to be a constant and it is defined as a measure of change in price over a given period or it is a measure of the rate of fluctuations in the price of a security over time. There are two major approaches of estimating volitality namely: historical and implified volitality. Historical volitality is defined as a measure of a stock's price movement on historical prices. whereas, implified volitality is a percentage that explains the current market price by reflecting the volitality that option traders expect for the return to the underlying asset during the life of an option Chiu [6]

2.1.10 Option pricing

Definition 6. According to Lishang Jiang [20], an option is referred to as an agreement that offers a holder to buy from or sell to, the seller of the option, an amount of an underlying asset sometimes referred to as financial securities. The buying and selling of assets can happen at a specific future time but the buyer, however, has no obligation to exercise the contract or an agreement. We define an exercise as buying or selling an asset according to the option contract or agreement. The specified price is called a strike price while the specified date is referred to as an expiration date.

The following definitions and types of an option by Lishang Jiang are useful in option pricing.

Types of options

1. Call option.
2. Put option.

Call option is defined as a contract to buy at a specified future time a certain amount of an underlying asset at a specified price. While, a put option is a contract to sell a certain amount of an underlying asset at a specified price in future time.

In option pricing, European and American options are referred to as terms of an exercise in a contract. European option is therefore defined as an option that can be exercised only proir to expiration date while an American option can be exercised on or prior to expiration date. Interesting examples and definitions are defined and explained in details on option pricing in [20] and references therein.

European option pricing and call-put assumptions by Lishang Jiang [20]

1. The market is arbitrage free.
2. All transactions are free of charge.
3. The risk-free interest rate is constant.
4. The underlying asset pays no dividends.

In general, if the price of an underlying asset at time T is S_T, the price of an option V_T, then there exist a function $V(S,t)$ such that $V_T = V(S_T,t)$.

Where $V(S,t)$ is deterministic fuction of tow variables.

We define call C_T and put P_T of an option in relation to the price of an underlying asset S_T and the expiry date $t = T$ as:

$C_T = (S_T - k)^+$ call option,

$P_T = (k - S_T)^+$ put option,

Where k denotes the strike price, T denotes expiration date and S_T denotes the price of an underlying asset at theexpiration date $t = T$

European put is considered as a solution of (1.1) with the following terminal and boundary conditions.

$P(s, T) = max(K - S, 0)$,

$P(0, t) = max(k - S, 0)$,

$P(s, t) \to 0, \;\; S \to +\infty$.

According to Ecem Herguner [16], For European call option initial and boundary condition of Black-Scholes PDF(1.1) for $s \in (0, \infty)$ and $t \in (0, T)$ is

$C(s, T) = max(S - K, 0)$,

$C(0, t) = 0$,

$C(s, t) \approx s$ and $x \to \infty$. where K is the strike price, T is the expiration date and $C(s, T)$ and $P(s, T)$ are the values of the option at time T when the option matures.

The above equations can be illustrated graphically as follows:

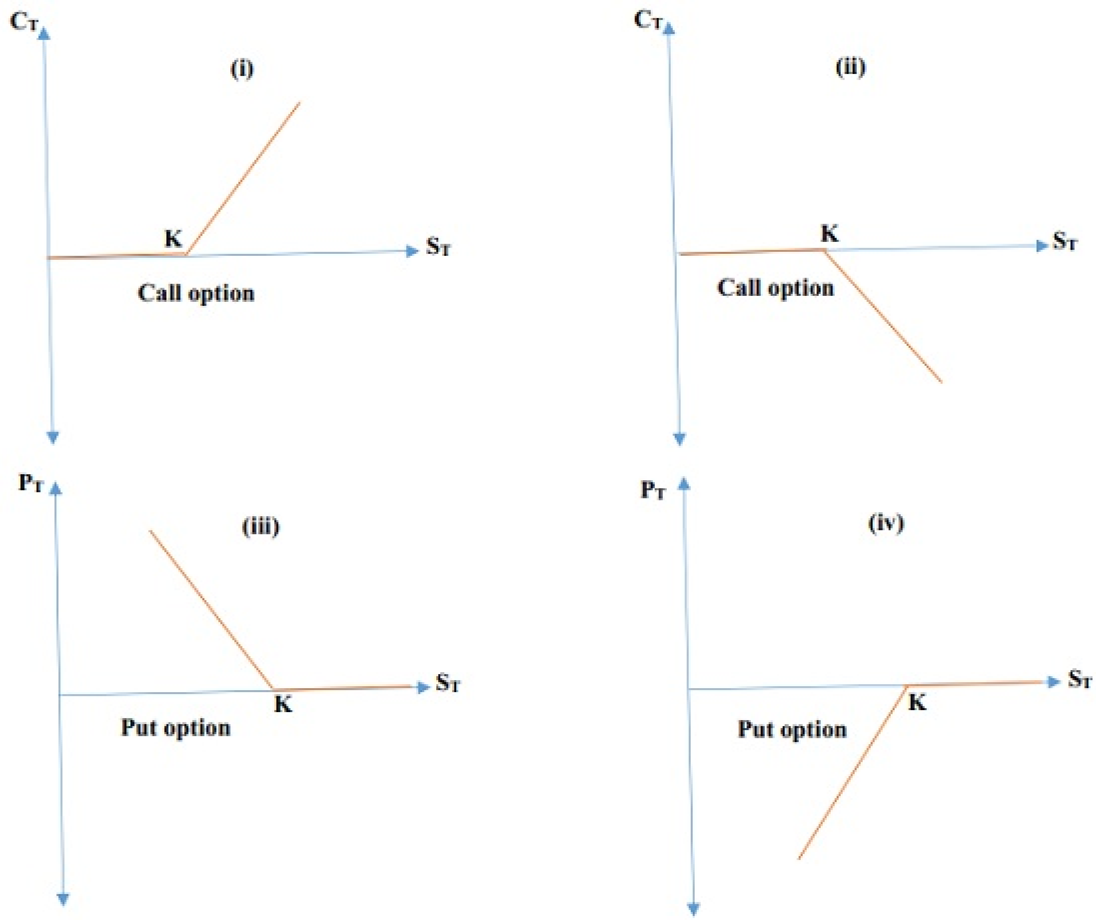

Figure 2.2: Call and Put options rough sketch demonstration

The above figure 2.2 represents the rough sketch option payoff where: (i) is a long call, (ii) is short call, (iii) is long put and (iv) short put. As per graphs above, since the buyer has no obligation to exercise the contract, options can be costing because one can make riskless profit by getting into long positions.

Common terms used in option Theory [41]

1. Premium.

The initial amount paid for the option.

2. Short position.

A negative amount of a quantity.

3. Long position.

A positive amount of a quantity.

4. At the money.

An option is at the money if the price of an underlying asset has leveled up close to the strike price.

5. Intrinsic value.

The payoff that would be received if the underlying asset is at its current level when the option expires.

6. In the money.

An option with a positive intrinsic value. A call option is in the money when the asset price is above the strike price.

7. Out of the money.

An option without any intrinsic value or with intrinsic value equal to 0. A call option is out of the money when the price of the underlying asset is below the strike price.

Chapter 3

Lie symmetries applied to fractional Black-Scholes model

This chapter is mainly on applying the information previously provided on chapter 2. We first compute the determining system of equations, then find the Lie algebras using properties provided in section 2.1.7. With the aid of some appropriate properties from section 2.1.5, we construct invariant solutions to our problem using the information provided in section 2.1.6.

3.1 Introduction

In this section, we employed equation (1.2) given by [39], then use group invariant, especially, Lie points symmetries where new complete Lie symmetry group and infinitesimal generators of the one dimensional fractional Black-Scholes pricing model are derived and then, the determining system of equations are obtained.
The value of European put option is taken as a solution of equation(1.2) with the following terminal and boundary conditions:

$u(x, T) = \max(k - x, 0)$,

$u(0, t) = kexp(-r(T - t))$,

$u(x, t) =\rightarrow 0 \quad x \rightarrow +\infty$,

where T, r, x represent expriration time, risk free interest and stock price respectively.

Determination of Lie symmetries

In this section, we make use of the fractional Black-Scholes equation (1.2)

$$\frac{\partial^\alpha u}{\partial t^\alpha} = (ru - rx\frac{\partial u}{\partial x})\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{\Gamma(1+\alpha)}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} \qquad t>0, 0<\alpha \leq 1$$

where r, σ and Γ denote the risk free interest rate, volatility and Gamma function respectively. $\frac{\partial^\alpha u}{\partial t^\alpha}$ by [12] denotes the Riemann-Lioville and it is defined as:

$$D_t^\alpha u(t,s) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{u(\xi,S)}{(t-\xi)^{\alpha+1-m}}\,d\xi & 0<m-1<\alpha \leq m, m \in N \\ \frac{\partial^n u}{\partial t^n} & \alpha = n \end{cases}$$

The vector field:

$$X = \xi^1(x,t,u)\frac{\partial}{\partial t} + \xi^2(x,t,u)\frac{\partial}{\partial x} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{3.1}$$

is the symmetry generator of (3.1) provided:

$$Pr^{\alpha,2}X\Big(\frac{\partial^\alpha u}{\partial t^\alpha} - \Big((ru - rx\frac{\partial u}{\partial x})\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\Gamma(1+\alpha)}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2}\Big)\Big)\Big|_{(1.2)} = 0 \tag{3.2}$$

Considering the Lie theory and imposing (1.2) into (2.15) we have the following invariant:

$$\phi_\alpha^0 + (rxu_x - ru)(1-\alpha)\frac{t^{-\alpha}}{\Gamma(2-\alpha)}\xi^1 + rx\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\phi^x + \frac{\Gamma(1+\alpha)}{2}\sigma^2 x^2 \phi^{xx} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}r\phi + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}ru_x\xi^2 + \Gamma(1+\alpha)\sigma^2 x u_{xx}\xi^2 = 0 \tag{3.3}$$

Now substituting (2.10),(2.11)and (2.17) into (3.3) yield the following:

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} + (\phi_u - \alpha D_t(\xi^1))\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \phi u}{\partial t^\alpha} + \sum_{n=0}^\infty \left[\binom{a}{n}\frac{\partial^\alpha \phi_u}{\partial t^\alpha} - \binom{a}{n+1}D_t^{n+1}(\xi^1)\right] D_t^{\alpha-n}u$$
$$\sum_{n=1}^\infty \binom{a}{n} D_t^n(\xi^2) D_t^{\alpha-n}u_x - (ru - rxu_x)(1-\alpha)\frac{t^{-\alpha}}{\Gamma(2-\alpha)} -$$
$$\frac{rxt^{1-\alpha}}{\Gamma(2-\alpha)}\left[\phi_x + u_x(\phi_u - \xi_x^2) - u_x^2\xi_u^2 - u_t\xi_x^1 - u_x u_t \xi_u^1\right] +$$
$$\frac{\Gamma(1+\alpha)\sigma^2 x^2}{\Gamma(2-\alpha)}[\phi_{xx} + u_x(2\phi_{xu} - \xi_{xx}^2) + u_{xx}(\phi_u - 2\xi_x^2) + u_x^2(\phi_{uu} - 2\xi_{xu}^2) - u_x^3\xi_{uu}^2 - \xi_u^1(u_t u_{xx} + 2u_x u_{xt})$$

$$- 3u_x u_{xx}\xi^2_u - 2u_{xt}\xi^1_u - u_t\xi^1_{xx} - 2u_x u_t\xi^1_{xu} - u_x^2 u_t\xi^1_{uu}]\frac{t^{1-\alpha}r\xi}{\Gamma(2-\alpha)} - \frac{t^{1-\alpha}ru_x\phi^2}{\Gamma(2-\alpha)} - \Gamma(1+\alpha)\sigma^2 x u_{xx}\xi^2 = 0 \quad (3.4)$$

Using Maple [33] and Maple Package FracSym routine FracDet and DESOLVEII 19 to get a system of determining equations and symmetry for the FDE of equation (1.2), we considered the value 2 as our forth argument for a calling sequence in FracSym worksheet package. The main purpose for this is to provide and obtain good balance between the information for the solution of determining equation and the speed at which they are obtained. The procedure and use of Maple code is shown in appendix 5

NOTE: The forth argument which can be an integer greater or equal to 1 is assigned to the number of terms to be “peeled off” from the sums which occur in the extended infinitesimal function for the fractional derivative 19 . Below is a list of determining equations which are linear and homogeneous PDEs where the subscript denotes the derivative with respect to the given variable:

$$\phi_{uu} = 0 \quad (3.5)$$

$$\alpha\xi^1_t = 0 \quad (3.6)$$

$$\alpha\xi^1_u = 0 \quad (3.7)$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^2_u = 0 \quad (3.8)$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^2_x = 0 \quad (3.9)$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^1_u = 0 \quad (3.10)$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^2_{uu} = 0 \quad (3.11)$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^1_{uu} = 0 \quad (3.12)$$

$$\alpha\xi^2_u(\alpha-1) = 0 \quad (3.13)$$

$$\alpha\xi^1_u(\alpha-1) = 0 \quad (3.14)$$

$$\alpha\xi^2_{tu}(\alpha-1) = 0 \quad (3.15)$$

$$\alpha\xi^1_{tu}(\alpha-1) = 0 \quad (3.16)$$

$$\alpha\xi^1_{uu}(\alpha-1) = 0 \quad (3.17)$$

$$\alpha\xi^2_{uu}(\alpha-1)=0 \tag{3.18}$$

$$\alpha\xi^1_{tt}(\alpha-1)=0 \tag{3.19}$$

$$\alpha\xi^2_{u}(\alpha-1)(\alpha-2)=0 \tag{3.20}$$

$$x^2\sigma^2\Gamma(\alpha+1)\xi^2_{u}(\alpha-1)=0 \tag{3.21}$$

$$\alpha\xi^2_{tu}(\alpha-1)(\alpha-2)=0 \tag{3.22}$$

$$\alpha\xi^2_{uu}(\alpha-1)(\alpha-2)=0 \tag{3.23}$$

$$\alpha(\xi^2_{tuu})(\alpha-1)(\alpha-2)=0 \tag{3.24}$$

$$\alpha(-\alpha\xi^2_{tt})+2\phi_{tu}+\xi^2_{tt}(\alpha-1)(\alpha-2)=0 \tag{3.25}$$

$$\alpha(\xi^2_{uuu})(\alpha-1)(\alpha-2)=0 \tag{3.26}$$

$$x\sigma^2\Gamma(\alpha+1)(\alpha x(\xi^2_t)-2\xi^1_x+2\xi^2)=0 \tag{3.27}$$

$$\frac{1}{\Gamma(2-\alpha)}\left[x(-\Gamma(2-\alpha)\Gamma(\alpha+1)\phi_{uu}\sigma^2x+2\Gamma(2-\alpha)\Gamma(\alpha+1)\xi^1_{ux}\sigma^2x+2t^{1-\alpha}\xi^1_u r)\right]=0 \tag{3.28}$$

$$(\alpha-1)\alpha(-\alpha\xi^2_{ttt})+3\phi_{tu}+2\xi^2_{ttt}=0 \tag{3.29}$$

$$\frac{1}{\Gamma(2-\alpha)}\left[x^2\sigma^2\Gamma(1+\alpha)\xi^2_{xx}\Gamma(2-\alpha)+2\alpha rut^{1-\alpha}\xi^2_u+2rxt^{1-\alpha}\xi^2_x\right]=0 \tag{3.30}$$

$$\frac{1}{\Gamma(2-\alpha)}x\left[-\Gamma(2-\alpha)\Gamma(\alpha+1)\xi^2_{ux}\sigma^2x+t^{1-\alpha}\xi^2_u\alpha r-t^{1-\alpha}\xi^2_u r\right]=0 \tag{3.31}$$

$$\frac{1}{\Gamma(2-\alpha)}[-2x^2\sigma^2(\Gamma(\alpha+1)\phi_{ux}\Gamma(2-\alpha)+x^2\sigma^2\Gamma(\alpha+1)\xi^1_{xx}\Gamma(2-\alpha)+2\xi^2rxt^{-\alpha}\alpha-2(\alpha)rxt^{1-\alpha}\xi^2_t$$
$$-2\xi^2rxt^{-\alpha}+2rxt^{1-\alpha}\xi^1_x-2\xi^1rt^{1-\alpha})]=0 \tag{3.32}$$

The Auxiliary conditions which include the sums and fractional derivative terms from this system are simply obtained using FracSym and the resulting equations are as follows;

$$\xi^2(x,0,u)=0 \tag{3.33}$$

$$\frac{1}{2t^\alpha\Gamma(2-\alpha)}[x^2\sigma^2\Gamma(\alpha+1)\frac{\partial^2}{\partial x^2}\phi t^\alpha\Gamma(2-\alpha)-2\alpha rut\frac{\partial}{\partial x}\xi^2-2u\frac{\partial^{\alpha+1}}{\partial t^2\partial u}\phi t^\alpha\Gamma(2-\alpha)+2\phi^2ru\alpha$$
$$+2\frac{\partial}{\partial u}\phi rut+2ruxt\frac{\partial}{\partial x}\phi+\frac{\partial^\alpha}{\partial t^\alpha}\phi t^\alpha\Gamma(2-\alpha)-2\xi^2ru-2\phi]=0 \tag{3.34}$$

$$\sum_{n=3}^{\infty}[-\frac{1}{n+1}\binom{\alpha}{n}D_{t^{\alpha-n}}uD_{t^{n+1}}\xi^2\alpha - D_{t^{\alpha-n}}uD_{t^{n+1}}\xi^2 n + D_{t^{\alpha-n}}\frac{\partial}{\partial x}uD_{t^n}\xi^1 n + D_t^{\alpha-n}\frac{\partial}{\partial x}uD_{t^n}\xi^1]$$

$$+\sum_{n=3}^{\infty}\big[\binom{\alpha}{n}\frac{\partial^{n+1}}{\partial t^n\partial u}\xi D_{t^{\alpha-n}}u\big] = 0 \quad (3.35)$$

Using DESOLVII to solve equations (3.5)-(3.32) we get the following infinitesimals for our FDE:

$$\xi^1 = xc_1, \tag{3.36}$$

$$\xi^2 = 0, \tag{3.37}$$

$$\phi = \beta(x,t) + uc_4. \tag{3.38}$$

Upon substitution of equations (3.36)-(3.38) into (3.34) and (3.35) to check whether auxiliary conditions are satisfied, we get:

$$\xi^2(x,0,u) = 0 \tag{3.39}$$

$$\frac{1}{2t^\alpha\Gamma(2-\alpha)}\big[x^2\sigma^2\Gamma(\alpha+1)\frac{\partial^2}{\partial x^2}(\beta(x,t)+uc_4)t^\alpha\Gamma(2-\alpha) - 2\alpha rut\frac{d}{dt}0 - 2u\frac{\partial^{\alpha+1}}{\partial t^\alpha\partial u}B(x,t)$$
$$+uc_4t^\alpha\Gamma(2-\alpha)+2\frac{\partial}{\partial u}(\beta(x,t)+uc_4)rut+2rxt\frac{\partial}{\partial u}(\beta(x,t)+uc_4)+2\frac{\partial^\alpha}{\partial t^\alpha}(B(x,t)+uc_4)t^\alpha\Gamma(2-\alpha)$$
$$-2(B(x,t)+uc_4)rt\big] = 0 \quad (3.40)$$

$$\sum_{n=3}^{\infty}\Big[-(\frac{1}{n+1})\Big[\binom{\alpha}{n}\big[D_{t^{\alpha-n}}(u(x,t))D_{t^{n+1}}(0)\alpha - (D_{t^{\alpha-n}}(u(x,t))D_t^{n+1}(0)n)$$
$$+D_{t^{\alpha-n}}(\frac{\partial}{\partial x}(u(x,t)))D_{t^n}(xc_1)n + D_{t^{\alpha-n}}(\frac{\partial}{\partial x}u(x,t))D_t^n(xc_1)\big]\Big]\Big]+$$
$$\sum_{n=3}^{\infty}\binom{\alpha}{n}\frac{\partial^{n+1}}{\partial t^n\partial u}(\beta(x,u)+uc_4)D_{t^{\alpha-n}}(u(x,t)) = 0 \quad (3.41)$$

Equations (3.40) and (3.41) evaluated yield (3.42) and (3.43) respectively

$$\frac{1}{2t^\alpha\Gamma(2-\alpha)}\big[x^2\sigma^2\Gamma(\alpha+1)\frac{\partial^2}{\partial x^2}(B(x,t))t^\alpha\Gamma(2-\alpha) - 2\alpha uc_4\mathbf{pochhammer}(1-\alpha,\alpha)t^\alpha$$
$$\Gamma(2-\alpha) + 2c_4rut + 2rxt\frac{\partial}{\partial x}B(x,t) + 2\big(\frac{\partial^\alpha}{\partial t^\alpha}(B(x,t)+uc_4\mathbf{pochhammer}(1-\alpha,\alpha)\big)$$
$$t^\alpha\Gamma(2-\alpha)+uc_4)rt\big] = 0 \quad (3.42)$$

$$
-\frac{1}{\alpha(\alpha-1)(\alpha-2)}6\binom{\alpha}{3}\Big(D_{t^n}(xc_1)\big(-\frac{\alpha^2}{2}+2^\alpha-\frac{\alpha}{2}-1\big)D_{t^{\alpha-n}}\big(\frac{\partial}{\partial x}u(x,t)\big)\Big) \\
+D_{t^{n+1}}(0)\big(D_{t^{\alpha-n}}(u(x,t))\big(-\frac{\alpha^3}{6}+2^\alpha-\frac{5\alpha}{6}-1\big)\big)\Big)=0, \quad (3.43)
$$

where **pochhemmer** is defined as rising or ascending factorial and in symbols, it is written as:

$$
\begin{aligned}
&(a)_0 = 0,\\
&(a)_1 = a,\\
&(a)_2 = a(a+1),\\
&(a)_3 = a(a+1)(a+2),\\
&\vdots\\
&(a)_n = a(a+1)(a+2)...(a+n-1),
\end{aligned}
$$

or

$$
(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \quad (3.44)
$$

The term $(a)_n$ follows from the hypergeometric series or functions.
We can easily see that (3.39) is satisfied, thus;

$$
0 = 0. \quad (3.45)
$$

Equation (3.42) and (3.43) are satisfied for total derivatives of $D_{t^{n_1}}$ and $D_{t^{1+n_1}}$ for some n_1. Therefore the final symmetries of our FDE is obtained as:

$$
\xi^1 = xc_1, \quad (3.46)
$$

$$
\xi^2 = 0, \quad (3.47)
$$

$$
\phi = \beta(x,t) + uc_4, \quad (3.48)
$$

where c_1, c_4 are arbitrary constants and $\beta(x,t)$ is an arbitrary solution of equation (1.2).
Now, imposing (3.46)-(3.48) into (3.1), we have the symmetry generator given by:

$$
X = xc_1\frac{\partial}{\partial x} + \big(\beta(x,t)+uc_4\big)\frac{\partial}{\partial u}. \quad (3.49)
$$

The symmetries are obtained by setting one constant to 1 and the rest to zero, so in vector form, equation (1.2) is spanned by the vector fields:

$$X_1 = x\frac{\partial}{\partial x}, \tag{3.50}$$

$$X_2 = u\frac{\partial}{\partial u}, \tag{3.51}$$

Infinite symmetry:

$$X_\infty = \beta(x,t)\frac{\partial}{\partial u} \tag{3.52}$$

Assumptions of the solution process

Upon using DESOLV11 package, the following non zero and linearly independent assumptions are obtained:

non-zero assumptions:

α, r, σ, t, x, t^α, $\alpha(\alpha-1)$, $t^\alpha\Gamma(2-\alpha)$, $\alpha(\alpha-1)(\alpha-2)$, $x\sigma\Gamma(\alpha+1)$, $x\alpha\ln(x)$, $\sigma x(\alpha-1)\Gamma(\alpha+1)$, $\alpha-2$, $\alpha-1$, $\Gamma(2-\alpha)$, $\Gamma(\alpha+1)$.

Linealy independent assumptions:

1,t,t^α

Obtaining Lie algebra

Below is a summury table Lie algebra obtained from commutators (3.50), (3.51) and (3.52) and we represent this in a form $[X_i, X_j]$ for some $i, j = 1, 2, \infty$:

Table 3.1: Lie algebras

$[X_i, X_j]$	X_1	X_2	X_∞
X_1	0	0	$x\beta_x\frac{\partial}{\partial u}$
X_2	0	0	$-\beta(x,t)$
X_∞	-$x\beta_x\frac{\partial}{\partial u}$	$\beta(x,t)$	0

Invariant solution

Using the information from section 2.1.6, we find the invariant solution.

Now consider the combination of X_1 and X_2 from equation (3.50) and (3.51) that is $X_1 + X_2 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$. With the use of the information on section 2.1.6, we have the following characteristic equation

Characteristic equation

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{u}. \tag{3.53}$$

From the second and third ratio on (3.53), we have that;

$$\begin{aligned} &\ln(u) - \ln(x) = \ln(c_1), \\ &\implies \ln\Big(\frac{u}{x}\Big) = \ln(c_1), \\ &\implies \frac{u}{x} = c_1, \\ &\implies u = xc_1, \end{aligned}$$

where c_1 is an arbitrary constant. From the first ratio on (3.53), we have that $t = c_2$, for some constant c_2.

$$\begin{aligned} \therefore u &= xg(c_2) \\ &= xg(t). \end{aligned} \tag{3.54}$$

To solve for $g(t)$, we substitute equation (3.54) into (1.2) and get the following results:

$$\frac{\partial^\alpha g(t)}{\partial t} = 0. \tag{3.55}$$

With the use of the information from [25], we now consider the Laplace transform Method from section 2.1.4 to solve for $g(t)$ on (3.55).

$$\begin{aligned} &\mathscr{L}\{\frac{\partial^\alpha g(t)}{\partial t}\} = 0 \\ &\implies s^\alpha G(S) - D^{-(1-\alpha)}y(0) = 0 \quad 0<\alpha \leq 1 \end{aligned} \tag{3.56}$$

Assume $D^{-(1-\alpha)}y(0)$ exists such that $D^{-(1-\alpha)}y(0) = c_3$, thus (3.56) can be written as

$$S^\alpha G(S) - c_3 = 0$$

$$\implies G(S) = \frac{c_3}{s^\alpha}. \tag{3.57}$$

Using the information from 2.1.4, we take the inverse Laplace transform of (3.57) to get

$$g(t) = \frac{c_3 t^{\alpha-1}}{\Gamma(\alpha)}. \tag{3.58}$$

Now, considering the symmetry X_1 given by (3.50), we have the following characteristic equation.

Characteristic equation

$$\frac{dt}{0} = \frac{dx}{x} = \frac{du}{0}. \tag{3.59}$$

The first ratio and the last ratio do not necessarily mean division by zero. Since t and u have similarity variable from (3.59), it suffice to write $u = g(t)$. Substitution of u into (1.2) yields:

$$D_t^\alpha g(t) = rg(t)\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}. \tag{3.60}$$

Solving for $g(t)$ from (3.60), the solution is trivial for $\alpha = 1$, that is

$$g(t) = ke^{rt} \text{ for some constant } k. \tag{3.61}$$

However, we are interested in finding the solution for all values of α such that $0<\alpha \leq 1$. Up to our knowledge and findings, equation (3.60) is not solvable.

Chapter 4

Results and interpretation.

4.1 Introduction

In this chapter, we give graphical solutions of option prices under the fractional Black-Scholes model of obtained solutions (3.58) and (3.61) using the same parameters for plotting and then plot the call and put option solutions within a given range. We finally make use of the results obtained from figure 4.1 to figure 4.9 and interpret how our work contributes in call and put option pricing in financial mathematics.

4.1.1 Graphical solutions

4.1.2 Graphical solution of $g(t) = \frac{c_3 t^{\alpha-1}}{\Gamma(\alpha)}$ in the xy-plane

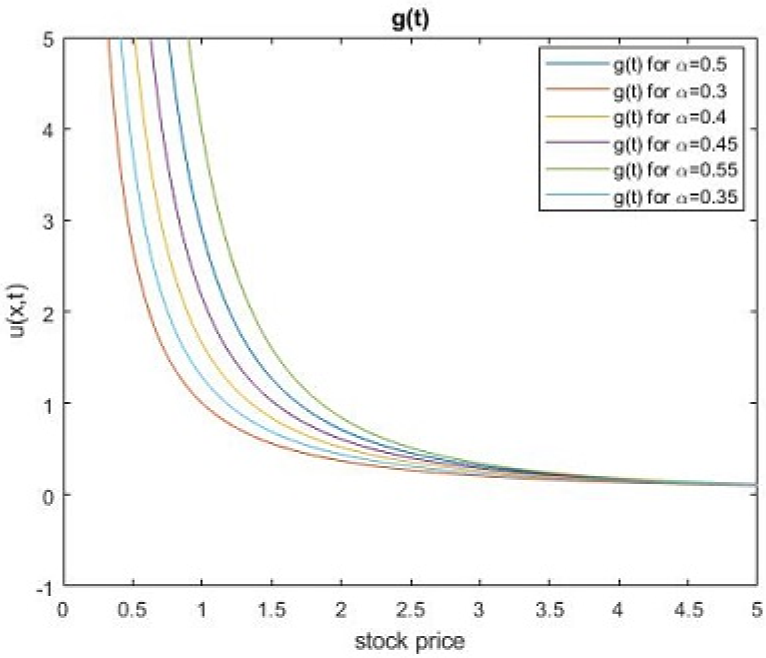

Figure 4.1: A 5x5 plot of (3.58) with $0<\alpha \leq 1$ on the xy-plane.

figure 4.1, shows graphical solution g(t) (3.58) obatined by Lie symmery method at $0<t<10$. Where $0<\alpha \leq 1$, $r = 0.01$, $\sigma = 0.01$ and stock price varies from 0 to 5.

The development trend of figure 4.1 is similar to that of a put option of a classical Black-Scholes equation(1.1) as well as the numerical solution graph in [39].

4.1.3 Graphical solution of g(t) (3.58) on 3D

Below is the graphical illustration of (3.58) in 3D with $\alpha = 0.5$.

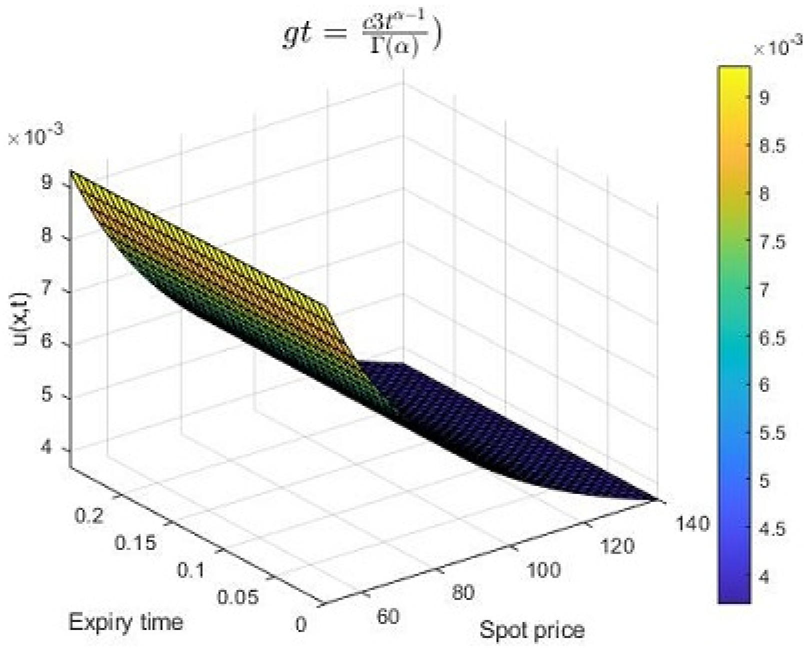

Figure 4.2: 60x60 solution of (3.58) with $0<\alpha \leq 1$ on 3D

4.1.4 Graphical solution of (3.61) in the xy-plane

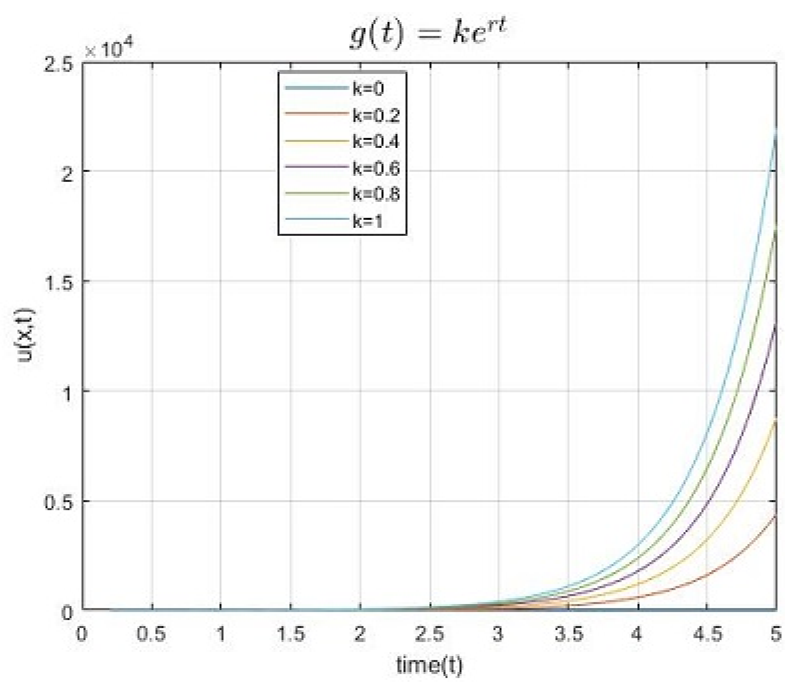

Figure 4.3: Solution of (3.61) with $r = 2$ on the xy plane.

4.1.5 Graphical solution of (3.61) on 3D-plane

Below is a 3D plot of equation (3.61).

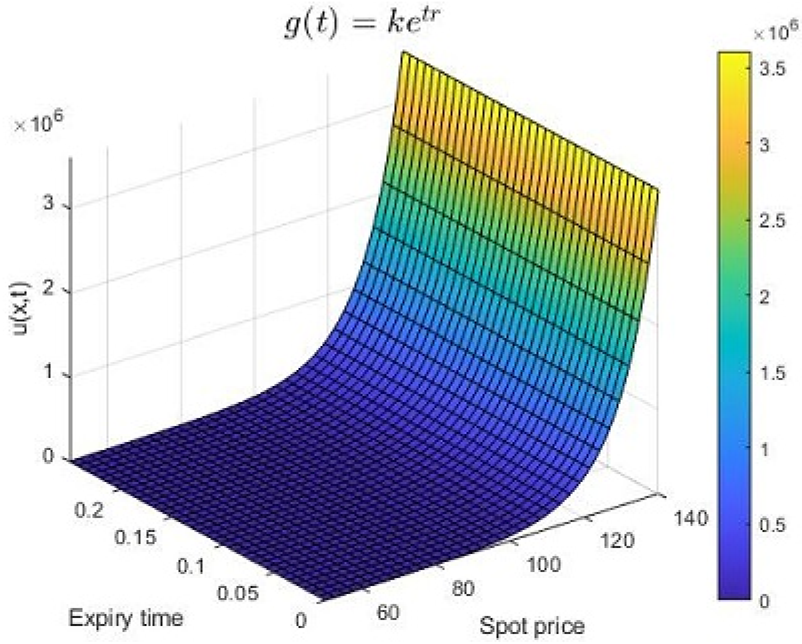

Figure 4.4: 60x60 solution of (3.61) with $r = 2$ on 3D

4.1.6 Call and Put option pricing plots

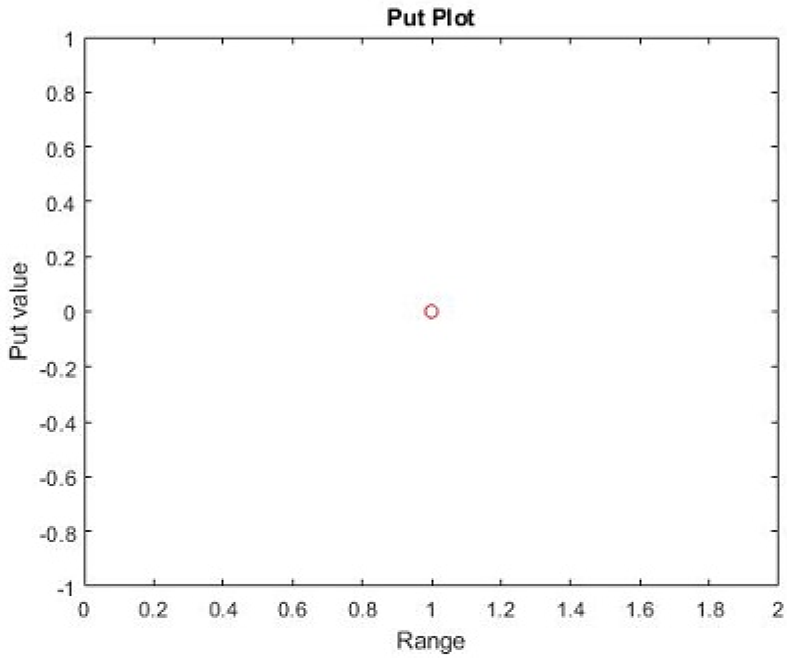

Figure 4.5: A put plot with $s = 10$, $\sigma = 0.1$, $r = 0.1$

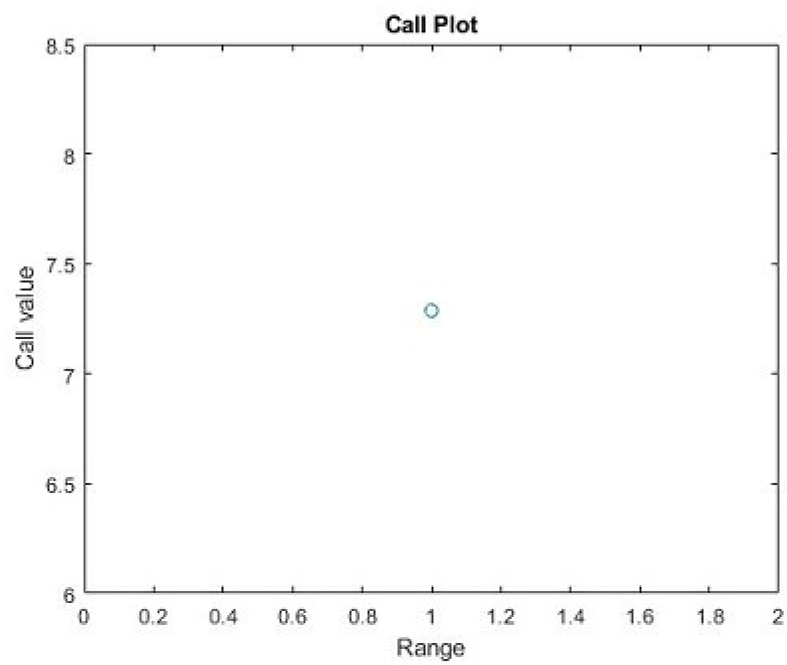

Figure 4.6: A call plot with $s = 10$, $\sigma = 0.1$, $r = 0.1$

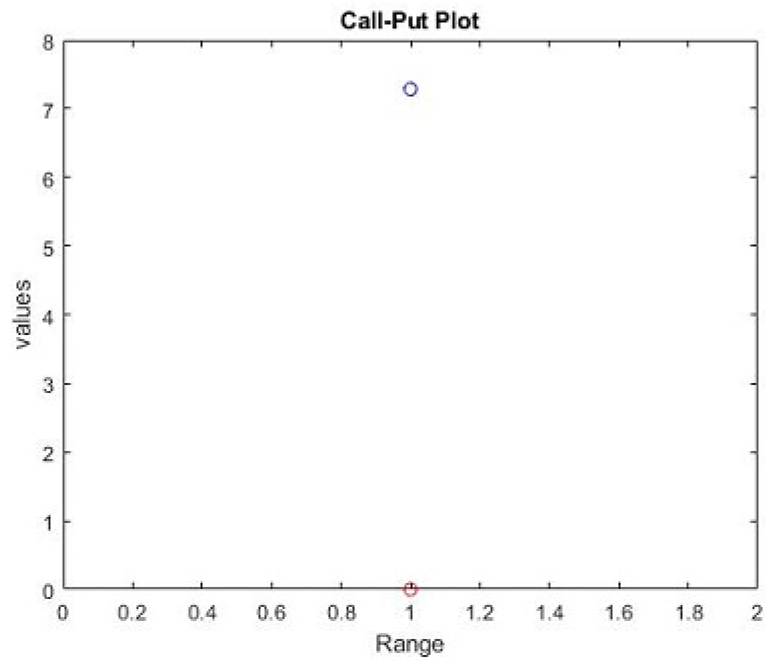

Figure 4.7: Call and Put plot with $s = 10$, $\sigma = 0.1$, $r = 0.1$

Now, considering a portfolio that invests α in the underlying asset and β in the risk free asset, the value of a portfolio is as follows
$P(0) = \alpha S(0) + \beta B(0)$.
Re-writting Black-Scholes formula as:

$$\frac{\partial u}{\partial t} + \frac{x^2\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} = ru - rx\frac{\partial u}{\partial x}, \tag{4.1}$$

the LHS of (4.1) above contains a time decay term, while the RHS is a riskless return. Using the definitions of call and put options in section 2.1.10, the put option will have the highest worth when the stock price is zero and as we can buy it at zero.
As per figure 4.1 to 4.7, the best place to put the stock will be when the stock worth is minimum, we see that from figure 4.7, put value saturates after long interval, at which the call value is very high, so to buy at this time could be cheaper.
Figure 4.8 and 4.9 below show a numerical solution of European and American put and call options of a Classical Black-Scholes model (1.1). The shape and development trend of this figures is similar to that of figure 4.1 and figure 4.3 respectively. The graphs of figure 4.8 and 4.9 below show one straight line and three curves. The dashed red straight line (together

with a part of the x axis) represents the payoff function of a European option with stock price = 60, rate of interest = 0.1, volitality = 0.5 and expiration time = 0.3. The orange curve represents the values of a standard European option, the blue curve a standard American option, and the green curve the perpetual (time-independent) option. In general, fractional Black-Scholes model seem to be more powerful in financial mathematics. Figure 4.8 and 4.9 were plotted using wolfram [26].

European and Americam call option

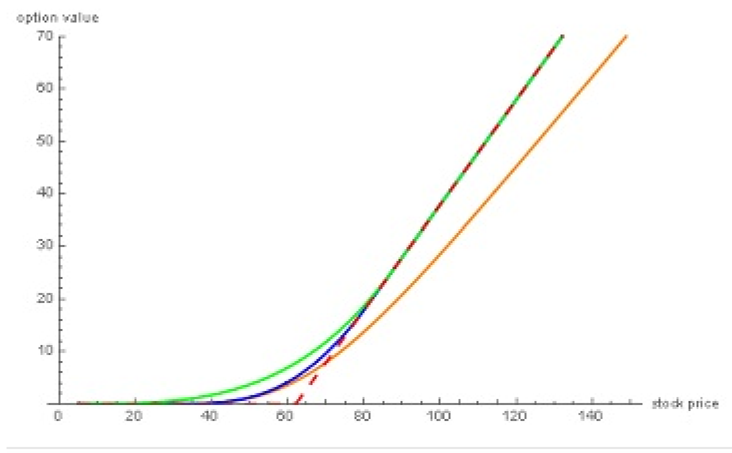

Figure 4.8: Call option plot with $s = 60$, $\sigma = 0.5$, $r = 0.1$

European and Americam put option

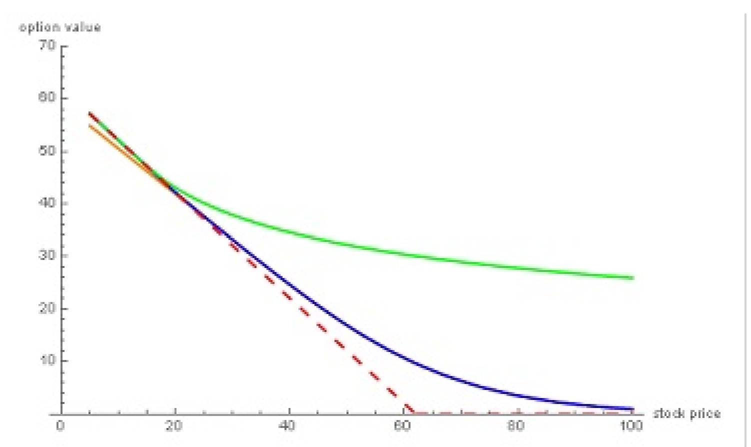

Figure 4.9: Put option plot with $s = 60$, $\sigma = 0.5$, $r = 0.1$

Chapter 5

Discussions and conclusions

In this work, Lie symmetry technique was implimented to solve time fractional Black-Scholes equation. Our objective was achieved since we were able to compute the determining equation which were obtained using Maple package fracSym. Moreover, upon solving each determining equation obtained, we constructed the infinitesimals which led to obtaining two invariant solutions. The invariant solutions obtained are constructed through the use of characteristic equation system. The style of solving characteristics equation in this discipline is not unique, so, we first considered the combination of X_1 and X_2 that is $X_1 + X_2$ which gave us solvable equation and hence fascinating results. However, considering X_1 alone gave us trivial solution when $\alpha = 1$ which reflected a good trend in relation to call option of a classical Black-Scholes (1.1), but our main concentration was on fractional order of derivatives, that is, we considered $0<\alpha \leq 1$. So for this interval $0<\alpha \leq 1$, the result obtained was not solvable. Considering the value of $\alpha = 1$ on equation (1.2) led to an underlying classical Black-Scholes equation (1.1). However, as per the graphs and results obtained, as we increases α, the risk free interest rate r increases as well. So, a decrease in α led to a decreace in the risk. In addition, when the stock price decreases, the risk also rises as volitality also rise when $\alpha<1$. This results in a negative skewness in stock returns, so, as we decrease α, local volitality rises and this indicates that x and u are negatively correlated to each other, which also results in negative skewness in stock return. In general, as we increase the stock price x, there is an increase in the leverage effect. Comparing solutions of g(t)

(3.58) obtained with numerical solution from [39] show how powerful fractional derivatives are. The successful application of Lie symmetry to solving fractional Black-Scholes equation proves how much effective and less computation our method is. However, further work is needed and will be of more interest on modeling other financial derivatives such as index return swaps, contract of difference and warrant.

Appendix

Definition 7. For complex number z with $\mathbb{R}^+$ the Gamma function is defined as:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \text{ for some } z \in \mathbb{R}^+.$$

The integral above converges absolutely for $\mathbb{R} > 0$. With the use of Gamma function formular, we have the following theorems.

Theorem 5.0.1. $\Gamma(z+1) = z\Gamma(z)$.

Proof. Let $z \in \mathbb{R}+$, for this proof we substitute the value $z+1$ where we see z on $\Gamma(z)$ and apply integration by part together with L'Hopitals rule.

$$\begin{aligned}\Gamma(z+1) = \int_0^\infty x^z e^{-x}dx &= -x^z e^{-x}\Big|_0^\infty + \int_0^\infty zx^{z-1}e^{-x}dx \\ &= \lim_{x\to\infty}(-x^x e^{-x}) - (o^z e^0) + z\int_0^\infty xz^{z-1}e^{-x}dx\end{aligned}$$

since $-x^z e^{-x} \to 0$ as $x \to 0$, then we have that

$$\begin{aligned}\Gamma(z) &= z\int_0^\infty x^{z-1}e^{-x} \\ &= z\Gamma(z)\end{aligned}$$

□

Theorem 5.0.2. $\Gamma(z) = (z-1)!$

Proof. Using the results obtained in **??**, the proof is as follows;

$$\begin{aligned}\Gamma(z) &= (z-1)\Gamma(z-1) \\ &= (z-1)(z-2)\Gamma(z-2) \\ &= (z-1)(z-2)...3\cdot 2\cdot 1\cdot \Gamma(1)\end{aligned}$$

$$= (z-1)!$$

It can be easly shown that $\Gamma(1) = 1$ by using definition 7. $\square$

This section consists of proofs , codes used to find plot figures, finding determining equations and infinitesimals.

Figure 1

Matlab Code

```
syms x
fplot(gamma(x))
grid on
title("Gamma function")
```

Figure 2

Matlab Code

```
clf
t=0:0.01:10;
s=t;
sigma=0.1;
r=0.1;
alpha=0.5;
alpha4=0.55;
alpha1=0.3;
alpha5=0.35;
alpha2=0.4;
alpha3=0.45;
c3=3;
T=1;
```

```
for i=length(alpha)
    y2=c3.*t.^(alpha(i)-1)./gamma(alpha(i));
    plot(y2,t,'DisplayName',['g(t) for \alpha=',num2str(alpha)])
    hold on
end
for i=length(alpha1)
    y3=c3.*t.^(alpha1(i)-1)./gamma(alpha1(i));
    plot(y3,t,'DisplayName',['g(t) for \alpha=',num2str(alpha1)])
    hold on
end
for i=length(alpha2)
    y4=c3.*t.^(alpha2(i)-1)./gamma(alpha2(i));
    plot(y4,t,'DisplayName',['g(t) for \alpha=',num2str(alpha2)])
    hold on
end
for i=length(alpha3)
    y5=c3.*t.^(alpha3(i)-1)./gamma(alpha3(i));
    plot(y5,t,'DisplayName',['g(t) for \alpha=',num2str(alpha3)])
    hold on
end
for i=length(alpha4)
    y6=c3.*t.^(alpha4(i)-1)./gamma(alpha4(i));
    plot(y6,t,'DisplayName',['g(t) for \alpha=',num2str(alpha4)])
    hold on
end
for i=length(alpha5)
    y7=c3.*t.^(alpha5(i)-1)./gamma(alpha5(i));
    plot(y7,t,'DisplayName',['g(t) for \alpha=',num2str(alpha5)])
    hold off
```

```
end
legend('location','best')
xlim([0 5])
xlabel('stock price')
ylim([-1 5])
ylabel('u(x,t)')
hold off
title('g(t)')
```

Figure 3

Matlab Code

```
clf
syms t d
c=sym(3);
S = sym(30);          % current stock price (spot price)
K = sym(95);           % exercise price (strike price)
alpha= sym(0.1);    % volatility of stock
T = sym(3/12);         % expiry tme in years
%r = sym(0.1);         % annualized risk-free interest rate

syms T S
C = c*T.^(alpha-1)./gamma(alpha);
fsurf(C,[50 140 0 0.25])
xlabel('Spot price')
ylabel('Expiry time')
zlabel('u(x,t)')
title('$g{t}=\frac{c3t^{\alpha-1}}{{\Gamma (\alpha})})$','
    Interpreter','latex', 'FontSize',16)
colorbar;
```

Figure 4 code

Matlab Code

```
clf%if needed, this clears the previous subplot
t=linspace(0.2,5,100);
r=2;
k=0:0.2:1;
for i=1:length(k)
     y=k(i)*exp(r*t);
plot(t,y,'DisplayName',['k=',num2str(k(i))])
hold on
end
legend('location','best')
xlabel('time(t)')%label x-axis
ylabel('u(x,t)')% label y-axes
grid on
title('$g(t)=ke^{rt}$','Interpreter','latex', 'FontSize',16)
```

Figure 5 code

Matlab Code

```
syms t d
k=sym(3);
S = sym(30);          % current stock price (spot price)
K = sym(95);           % exercise price (strike price)
alpha= sym(0.1);    % volatility of stock
T = sym(3/12);         % expiry tme in years
r = sym(0.1);         % annualized risk-free interest rate
```

```
syms T S
C = k*exp(T*r);
fsurf(C,[50 140 0 0.25])
xlabel('Spot price')
ylabel('Expiry time')
zlabel('u(x,t)')
title('$g(t)=ke^{tr}$','Interpreter','latex', 'FontSize',16)
colorbar;
```

Figure 6,7 and 8

Matlab Code

```
s = 10;
k=3;
T=1;
sigma=0.1;
r=0.1;
q=0;
[c,p] = BSM(s,k,T,sigma,r,q);

figure(1)
plot(c,'o')
title("Call Plot")
ylabel('Call value')
xlabel('Range')

figure(2)
plot(p,'ro')
title("Put Plot")
```

```
ylabel('Put value')
xlabel('Range')

figure(3)
plot(c,'bo')
hold on
plot(p,'ro')
title("Call-Put Plot")
ylabel('values')
xlabel('Range')

function [c,p]=BSM(s,k,T,sigma,r,q)
d1=sigma*sqrt(0.5)\(log(s/k) + (r-q+sigma^2/2)*T);
d2 = d1 - sigma*sqrt(T);
c = s*exp(-q*0.5)*normcdf(d1)-k*exp(-r*T)*normcdf(d2);
p = k*exp(-r*T)*normcdf(-d2)-s*exp(-q*T)*normcdf(-d1);
X = ['Call: ',num2str(c)];
Y = ['Put: ',num2str(p)];
disp(X);
disp(Y);

end
```

Below is how we implimented and ran the code to find the determining equations of (1) and its infifnitesimal. This is done in maple software using FracSym package [19, 33]

```
restart;
Instructional workheet for the FracSym package
G. F. Jefferson and J. Carminati
Read in accompanying packages: ASP, DESOLVII and initialise using the with command:
```

```
read ‘ASP v4.6.3.txt‘:
DESOLVII_V5R5 (March 2011)(c), by Dr. K. T. Vu, Dr. J.
Carminati and Miss G. Jefferson
The authors kindly request that this software be referenced, if it is used
in work eventuating in a publication, by citing the article:

K.T. Vu, G.F. Jefferson, J. Carminati, Finding generalised
symmetries of differential equations
using the MAPLE package DESOLVII,Comput. Phys. Commun. 183
(2012) 1044-1054.
-------------
ASP (November 2011), by Miss G. Jefferson and Dr. J. Carminati
The authors kindly request that this software be referenced, if
it is used in work eventuating in a publication, by citing
the article:
G.F. Jefferson, J. Carminati, ASP: Automated Symbolic
Computation of Approximate Symmetries
of Differential Equations, Comput. Phys. Comm. 184 (2013)
1045-1063.


with(ASP);
[ApproximateSymmetry, applygenerator, commutator]
with(desolv);
[classify, comtab, defeqn, deteq_split, extgenerator, gendef,
genvec, icde_cons, liesolve, mod_eq, originalVar, pdesolv, reduceVar,

reduceVargen, symmetry, varchange]

Read in FracSym and initialise using the with command:
```

```
read ‘FracSym.v1.16.txt‘;
FracSym (April 2013), by Miss G. Jefferson and Dr. J. Carminati

The authors kindly request that this software be referenced, if it is used in

work eventuating in a publication, by citing:

G.F. Jefferson, J. Carminati, FracSym: Automated symbolic
computation of Lie symmetries of fractional differential equations, Comput. Phys. Comm.

Submitted May 2013.

with(FracSym);
[Rfracdiff, TotalD, applyFracgen, evalTotalD, expandsum, fracDet,

fracGen, split]
```

BASIC OPERATORS

The Riemann-Liouville fractional derivatives is expressed in "inert" form using the
FracSym routine Rfracdiff.
The explicit formula for the form of these fractional derivatives may be found in I.
Podlubny, Fractional differential equations: An introduction to fractional derivatives,

```
Rfracdiff(u(x, t),t,alpha);
```

If the fractional derivative is taken for a product, the generalised Leibnitz rule
is used to express the result (the product operator used is &* and is non-commutative).

```
Rfracdiff(u(x, t)&*v(x,t),t,alpha);
```

Fractional derivatives of integer order revert to the MAPLE diff routine.

```
Rfracdiff(u(x, t)&*v(x,t),t,2);
```

The FracSym rouine TotalD may also be used to find total derivatives. evalTotalD is then used to evaluate the result (in jet notation). For example,

```
TotalD(xi[x](x, y),x,2);
```

```
fde1:=Rfracdiff(u(x,t),t,alpha)=r*u*((t^(1-alpha))/Gamma(2-alpha))-r*x*((t^(1-alpha))
/Gamma(2-alpha))*(diff(u(x,t),x))-((x^2)*(sigma^2)*((Gamma(1+alpha))/2)*(diff(u(x,t),
x,x))); t>0;
```

We use the the FracSym routine fracDet to find the determining equations for the symmetry for fde1.
NOTE: The fourth argument (some integer at least 1) corresponds to the number of terms to be "peeled off" from the sums which occur in the extended infintesimal function for the fractional derivative. A value of 2 provides a good balance between information for

```
deteqs:=fracDet([fde1],[u],[x,t],2);
Intervals/values considered for the fractional derivative/s:

{0 < alpha, alpha < 1}

n1:=nops(%[1]);
n1 := 29
```

The output is as follows: the first list contains the determining equations which are linear, homogeneous PDEs. The second list contains "auxiliary conditions" which include sums and fractional derivative terms. The third list contains the infinitesimal functions to be solved. The fourth list contains all variables. We may solve the first list using the DESOLVII pde solver:

```
sol1:=pdesolv(expand(deteqs[1]),deteqs[3],deteqs[4]);
```

We check that these solutions satisfy the auxiliary conditions:

```
subs(sol1[3],deteqs[2]);
```

```
value(%);
```

The second and third conditions are satisfied for total derivatives
"D[t^n1], D[t^(1 + n1)]"

The first condition implies:

```
subs(t=0,sol1[3]);
rhs(%[2])=0;
0=0
```

Hence, the final symmetry for the FDE is:

```
subs(%,[sol1]);
```

Which can be expressed in vector form using DESOLVII's genvec routine as

```
genvec(%[3],%[4],[x,t,u]);
classify();
```

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