Lie symmetry analysis of partial differential equations associated with the Westervelt model in non-linear acoustics

by

Theko Martin Sekhesa (201702343)

The full dissertation is submitted for the degree of Master of Science in Mathematics (Applied) in the Department of Mathematics and Computer Science.

Supervisor: Mr. Ngaka John Nchejane

Co-supervisor: Dr. Wetsi David Poka

Declaration

I, Theko Martin Sekhesa, declare this dissertation entitled, "Lie symmetry analysis of partial differential equations associated with the Westervelt model in non-linear acoustics" represents my own work, which has been done for the degree of Master of Science in Mathematics (Applied). I have read the University, faculty, and department regulations, and I accept the responsibility for the conduct of the procedures in accordance with University regulations, and where I have consulted the published work, this has been clearly attributed observing the University policy on plagiarism.

T.M.S., SEKHESA

(Initials, Surname)

27th June 2024

(Date)

Abstract

Sound-wave propagation in compressible media has significant non-linearities. It is for this reason, that modelling such phenomena necessitates a partial differential equation that considers these non-linear effects. That is where the Westervelt model comes into action, it is a non-linear partial differential equation used to model the propagation of high finite amplitude sound waves in non-linear acoustics, i.e., sonar systems, medical ultrasound imaging and non-destructive testing. The propagation of such waves takes place in non-linear media that exhibit thermal and viscous characteristics, e.g. human tissue.

Two equations that represent the Westervelt model are considered in this work, the first one is the usual equation that has the dissipative term as the third-order temporal derivative. The second equation is where the linear wave relation has been inserted for the dissipative term.

Symmetry analysis is performed on each of the models individually. This involves generating an over-determined system of linear homogeneous partial differential equations, which is solved to get the Lie point symmetries. Then, with the aid of the adjoint and commutator tables, an optimal system of sub-algebras is found and used in the similarity reductions to get the invariant sub-models. One-parameter Lie point groups are constructed, followed by exact invariant solutions for the sub-models, with the modified simple equation method applied to find some solitary wave solutions. Lastly, simulations in terms of 2D and 3D graphs representing the invariant solutions are presented.

Keywords

Lie symmetry analysis;

Lie point symmetry;

One-parameter Lie group;

Partial differential equations;

Westervelt model;

Invariant solution;

Non-linear acoustics.

Dedication

To my family and friends.

"*I wanted to do Mathematics, not change the world*"

-Karen Uhlenbeck

Acknowledgments

I would like to express my heartfelt gratitude to the National University of Lesotho, under the Department of Mathematics and Computer Science, for having granted me the opportunity to further my studies. Also, I would like to extend my gratitude to the National Manpower Development Secretariat, for the financial assistance in the form of tuition, accommodation fees and a living wage throughout this en devour. Moreover, I am immensely thankful to my supervisors, Mr. Ngaka John Nchejane and Dr. Wetsi David Poka, for their everlasting guidance and support. Lastly, I would like to appreciate my lecturers, friends, classmates and family for believing in me.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The science of sound wave propagation and its interaction with matter is known as acoustics. In particular, acoustics is concerned with the transfer of acoustical energy from one place to another in the form of mechanical waves [42]. This influences the particles inside the medium to vibrate giving off what is known as sound. Therefore, it is evident that for acoustic propagation to take place, a physical medium should be present because sound waves cannot travel in a vacuum. One of the major physical dynamics that is utilised in describing sound propagation is acoustic pressure, defined as the amplitude level of sound at a specific location in space [32].

Although it is assumed that sound wave propagation is a linear process, there are instances where linear assumptions can be neglected in deriving acoustic models [20, 32], thus results in complicated non-linear models in the form of partial differential equations (PDEs). It has been the goal of scientists and engineers to find solutions to the said models [21, 39, 82], which enables educated predictions concerning acoustic propagation in various media.

We start our exploration by delving into the field of non-linear acoustics, we review the field's evolution, consider a brief derivation of the Westervelt model and the contributions of researchers. Additionally, we provide a concise overview of the life and achievements of the inventor of Lie symmetry analysis. Finally, we analyse the relevant literature related to the model of interest in this study.

1.1 Background

1.1.1 Non-linear acoustics

Non-linear acoustics is a well-developed branch of science and technology that is concerned with the investigation and application of large amplitude sound waves to real-life problems using non-linear models [56, 78]. These have caught the interest of numerous scientists as early as the seventeenth century because a majority of phenomena that occur in real life have some inherent non-linearity, as such, they provide a comprehensive understanding of the system behaviour. As early as the seventeenth century, it was established through experimentation that sound exhibits fundamental wave properties like reflection, refraction, diffraction, and interference [7, 75]. In the eighteenth century, Leonard Euler discovered a connection between the theory of sound propagation and continuum physics [33, 59]. As a result, it was possible to apply D'Alembert's equation in fluid mechanics and elasticity. There are still some acoustical phenomena that could not be explained by the use of D'Alembert's equation, or modelling acoustics using D'Alembert's equation came to be known as linear acoustics [27, 30]. Among others, it was dealing with significantly large finite amplitude sound waves where linear acoustics failed.

Non-linear acoustics began its development as a scientific discipline in the field of acoustics in the mid-nineteenth century [27]. Nonetheless, advancements in non-linear acoustics had already been made in the late eighteenth century. Leonard Euler derived the non-linear acoustic plane wave equation in the air assuming that air had the characteristics of Boyle's law [36, 72]. Advancements in solving non-linear wave equations were made by Lagrange and Poisson [27]. However, the development of non-linear wave propagation is attributed to G.B. Airy for his work in 1849 [5], S. Earnshaw between 1858 and 1860 [35] and B. Riemann in 1860 [54]. Another important stride in non-linear acoustics was the description of a wave after the formation of a shock [10]. The main difficulty was discontinuities associated with non-linear waves propagating in lossless fluids. Imperative impact on this issue was due to the works of Rankie in 1870 and Hugoniot from 1887 to 1889 [79], they formulated conservation of mass, momentum, and energy, linking the flow field behind a shock and a flow field ahead of it.

By 1935, Fubini-Ghiron had made substantial progress in the theory of finite wave propagation in non-viscous fluids [13]. The successful inclusion of viscosity in the description of the weak shock formation and propagation was achieved earlier by Lord Rayleigh and G.I. Taylor in 1910 [31]. It was around 1948 that the standard non-linear acoustic plane wave equation began to be used, this consequently led to the derivation of the fundamental equations that describe sound propagation in non-linear regimes [22, 87]. For a thorough history concerning the field of non-linear acoustics refer to [9, 76] and the references therein.

The three fundamental equations governing the propagation of acoustic waves in non-linear fluids are;

- Westervelt Equation [90]

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 p}{\partial t^3} + \frac{\beta}{\rho c_0^2}\frac{\partial^2 p^2}{\partial t^2} = 0, \tag{1.1}$$

- Burger's Equation [14]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1.2}$$

- Khokhlov-Zabolotskaya-Kuznetsov (KZK) Equation [95]

$$\frac{\partial^2 p}{\partial z \partial \tau} - \frac{c_0}{2}\nabla_{\perp}^2 p - \frac{\delta}{2c_0^3}\frac{\partial^3 p}{\partial \tau^3} - \frac{\beta}{2\rho_0 c_0^3}\frac{\partial^2 p^2}{\partial \tau^2} = 0. \tag{1.3}$$

Equations (1.1) to (1.3) describe the propagation of sound waves in fluids, meaning that the constitutive equations of fluid mechanics are utilised in their derivation. Practical uses of non-linear acoustics include shock waves, acoustic cavitation, high intensity focused ultrasound (HUIF), e.t.c. (see [29, 88, 94]).

The constitutive equations for non-linear acoustics are grouped in two, that is, propagation in fluids and propagation in solids. This work focuses on the former, the fundamental equations of fluid mechanics that are used in the derivation of the primary equations of non-linear acoustics in fluids are (see [27]):

1. Constitutive equation of non-linear acoustics in fluids

$$\frac{\partial \rho_a}{\partial t} + \rho_0 \nabla \cdot \vec{v_a} = -\rho_a \nabla \vec{v_a} \cdot \nabla \rho_a. \tag{1.4}$$

2. Equation of motion

$$\rho_0 \frac{\partial \vec{v_a}}{\partial t} + \nabla p_a - \left(\xi + \frac{4}{3}\eta \right) \triangle \vec{v_a} = -\nabla L. \tag{1.5}$$

3. Equation of State

$$\rho_a = \frac{p_a}{c_0^2} - \frac{B}{2A}\frac{p_a^2}{\rho_0 c_0^4} + \frac{\kappa}{\rho_0 c_0^4}\left(\frac{1}{c_v} - \frac{1}{c_p} \right)\frac{\partial p_a}{\partial t}. \tag{1.6}$$

where κ is the thermal conductivity. Parameters c_v and c_p are the specific heat capacities per unit mass at constant volume and pressure, ξ is the bulk viscosity, η is the shear viscosity and c_0 is the speed of sound. The pressure, density and velocity are decomposed into their time-mean and fluctuating components as follows,

$$p = p_0 + p_a, \qquad \rho = \rho_0 + \rho_a, \qquad \vec{v} = \vec{v_0} + \vec{v_a}.$$

The quantities A and B, are found from the Taylor expansion of the deviation $p - p_0$, from the time-mean pressure p_0, corresponding to the density deviation $\rho - \rho_0$, at constant entropy. The parameter L is the Lagrangian density,

$$L = \frac{1}{2}\rho_0 \vec{v}_a^2 - \frac{p_a^2}{2\rho_0 c_0^2},$$

and ∇ is the del operator,

$$\nabla = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}.$$

Also, the Laplacian operator is given by $\triangle$, such that

$$\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Substituting eq. (1.6) into eq. (1.4), assuming the time-mean density ρ_0 to be constant and assuming the velocity fluctuation term, $\vec{v}_a$, is negligible due to the smallness of the time dependence of the wave at a given distance from sound source caused by the non-linearities and dissipation effects, we get the following non-linear propagation equation

$$\triangle p_a - \frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 p_a}{\partial t^3} = -\left(\frac{\beta}{\rho_0 c_0^4}\right)\frac{\partial^2 p_a^2}{\partial t^2} - \left(\triangle + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)L, \tag{1.7}$$

where $\beta = 1 + \frac{B}{2A}$ is the non-linearity parameter and $\delta = \left(\frac{1}{\rho_0 c_0^2}\right)\left(\xi + \frac{4}{3}\eta\right) + \left(\frac{\kappa}{\rho_0 c_0^2}\right)\left(\frac{1}{c_v} - \frac{1}{c_p}\right)$ is sound diffusivity. Thus, by dropping the last term on the right hand side of eq. (1.7), this is done because we assume local non-linear effects modelled by the quadratic velocity term are negligible, we get

$$\triangle p_a - \frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 p_a}{\partial t^3} = -\left(\frac{\beta}{\rho_0 c_0^4}\right)\frac{\partial^2 p_a^2}{\partial t^2}. \tag{1.8}$$

Equation (1.8) is known as the Westervelt equation[1], which models the propagation of sound waves in a compressible medium [89]. The analytical solutions of the Westervelt equation are used in parametric acoustic array and in medical ultrasound, or they can be used as benchmarks for testing numerical schemes of physical experiments [23, 51].

Recall the linear wave equation relation,

$$\triangle p_a = \frac{1}{c_0^2}\frac{\partial^2 p_a}{\partial t^2} \implies c_0^2 \triangle \left(\frac{\partial p_a}{\partial t}\right) = \frac{\partial^3 p_a}{\partial t^3}.$$

Substituting the above relation into the Westervelt model eq. (1.8) for the dissipative term, yields, a modified version of the Westervelt equation.

---

[1] Named after Leroy R. Westervelt whose work paved the way for understanding how sound waves behave under non-linear conditions [48].

In this dissertation we study both versions the (1 + 1)-dimensional dissipative Westervelt models.

- Model I

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 u}{\partial t^3} + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0,$$

- Model II

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0,$$

where it is assumed that the direction of propagation is specified in the x-coordinate and t represents time. Acoustic pressure is given by $u(x, t)$ and sound diffusivity is δ. The third-order derivatives in both models represent the dissipation or loss term which is a result of thermal conduction, in a sense that some of the acoustical energy is lost due to conversion into heat energy, and fluid viscosity.

1.1.2 Historical preface to Lie group theory

The information provided below and more can be found in [15, 40, 53, 77] and the subsequent references.

Born in the vicarage at Eid in Nordfjord, Norway, on December 17, 1842, Marius Sophus Lie was the son of a rector at the parish of Eid. For most of his juvenile years, he was heavily influenced to study to be a parish minister by his father, but later in his early twenties, he began to take an interest in mathematics. In 1869, he travelled to Berlin and met with Felix Klein, thus their lifelong friendship and the academic relationship began.

In 1870, during their time in Paris, they collaborated on W-curves. These curves are homogeneous, meaning that no point on them differs from another [3]. Essentially, W-curves can be visualized as either straight lines or circles. The concept of curve homogeneity implies the existence of isometries that transform the curve into itself, mapping each point to other points on the same curve. For a straight line, this self-isometry group is the group of translations along the direction of the line, for a circle, it is the group of rotations and for a point P on a logarithmic spiral whose equation in polar coordinates (r, θ), is $r = a^{\theta}$, where r is the distance from the origin a, to point P. Other self-similar transformations are $\tilde{r} = a^{\tilde{\theta}} \cdot r$, and $\tilde{\theta} = \theta + c$, some arbitrary constant c. They transform the point (r, θ) into itself and the spiral onto itself.

Lie worked closely with Klein on finding each curve on the plane that has a group of projective transformations that map the curve into itself, they called such curves W-curves. This research influenced Lie into studying one-parameter subgroups of the group of projective transformations which consequently would lead him to the construction of Lie algebras. Additionally, he established the idea of an infinitesimal transformation and discovered the concept of a contact transformation. Contact transformations generalize surface mappings in a space that includes points and their tangents [81].

After Lie's collaboration with Klein in Berlin and Paris came to an end, he devoted the remainder of his life studying continuous groups known as Lie groups, and their relationship to differential equations [11, 28, 86]. The adjective "continuous" here, emphasizes that the transformations can be continuously changed by slight alterations of the parameter determining the particular element of the group. One of the crucial points of Lie's theory was that one could assign to each continuous group a much simpler algebraic object, its Lie algebra.

From 1892 to 1893, the Kazan Physico-Mathematical society created the international Lobachevsky prize, the first prize was awarded in 1898 to Sophus Lie. He died on February 18, 1899. A committee devoted to publishing his mathematical works was formed in 1900.

1.1.3 Literature review

Various analytical solution methods have been developed to solve non-linear partial differential equations, to name a few, Lie group analysis [2, 4, 11, 16–19, 25, 28, 41, 43–46, 50, 62, 69–71, 86], Kudryashov method [49], tanh-function method [97], Bernoulli functional method [24] and (G'/G)-expansion technique [92]. Some of the studies carried out on the Westervelt model are found in references [52, 66, 80, 85, 99]. Yu.A. Chirkunov carried out a study in [18] where Lie symmetry analysis was used to obtain the invariant sub-models of the (3+1)-dimensional Westervelt equation. Following this, some of the analytical solutions to the those invariant sub-models were found, and can be used as test solutions against numerical methods. In [17], the Lie point symmetries of the lossless Westervelt equation were studied and used for similarity reductions to obtain invariant sub-models. The invariant solutions to these sub-models were determined analytically and used to study intensive acoustic-waves for which pressure and its derivative were specified in the direction of one axis at initial time t_0.

Symmetries and conservation laws are inherent in PDEs as shown by Noether's theorem, by investigating symmetries, it is possible to derive conservation laws for different dynamical systems [12]. In [2], Lie symmetry analysis is conducted on the Westervelt equation with dissipation, the Lie point symmetries are determined and employed to derive non-local conservation laws by exploiting the hidden variational structure of the model.

Sidra Ghazanfar et al. in [34] used the Kudryashov and modified Kudryashov methods to find the exact solutions of the (1+1)-dimensional Westervelt model corresponding to Model II. The obtained solitary wave solutions are discussed with the help of surface and contour plots. It is shown that the solutions represent high-amplitude ultrasound waves used in medical imaging and ultrasonic therapy. The confirm-able time-fractional Westervelt equation was studied by Tahira S. Shaikh et al. in [84]. They employed the modified exponential rational functional method and the modified (G'/G)-model expansion method to find analytical solutions of the fractional Westervelt model. Different hyperbolic, exponential, periodic and plane wave function solutions are found. The exact solutions are shown to characterise wave structures that are employable in studying high-frequency sound wave propagation.

The third-order non-linear fractional Westervelt equation was investigated by Usman Younas et al. in [93]. Various wave structures were extracted in the form of solitary solutions by the use of a new extended direct algebraic approach. The obtained solutions are useful in ultrasound propagation in tissue structures, underwater acoustics e.t.c. New soliton solutions to the M-fractional Westervelt equation were found by Haitham Qawaqneh et al. in [74]. The solutions are found by utilizing $\exp_a$ function and modified simplest equation technique. The found solutions include dark, periodic, dark-bright and other solitons which can be of importance in medical imaging and ultrasonic therapy.

So far, only manuscripts concerned with analytical approaches to solve the Westervelt model have been reviewed, but, an alternative approach used by other researchers is approximating the solution using numerical methods. Guy V. Norton and Robert D. Purrington in [66] realized that the Westervelt equation with the traditional loss term fails to encompass the dispersive traits of the medium. By replacing the loss term with a causal convolutional propagation factor and solving the model using the method of finite differences, it is shown that the time propagation factor imparts the correct phase and attenuation to the signal. Similarly, Katherine Baker et al. in [6] represented the attenuation with a class of non-local time operators while studying the Westervelt model with time-fractional damping. They utilised a hybrid scheme comprising of the trapezoidal rule and the quadrature method coupled with a piecewise linear Galerkin finite element space discretization. Find more numerical approaches to solving the Westervelt model in [37, 38, 65, 67, 83].

This dissertation is partly similar to [2, 18], in that, the models of interest in the aforementioned have the dissipation term described by the third-order temporal derivative of acoustic pressure. Likewise, the obtained Lie point symmetries are the same as the ones found in this work, with the distinction that in [18], the space translation symmetries include additional spatial coordinates, y and z. In contrast, the partial differential equation given by Model I has physical acoustical parameters unlike in [18], where transformations of the independent and dependent variables, $t = \frac{\rho_0 c_0^2}{b} c_0 T$, $x = \frac{\rho_0 c_0}{b} \mathbf{x}'$ and $p(t, x) = \frac{\epsilon}{\rho_0 c_0^2} P(T, \mathbf{x}')$, such that $P(T, \mathbf{x}')$ is the acoustic pressure, T is the time, $\mathbf{x}' = (x', y', z') \in \mathbb{R}^3$, ϵ is the non-linearity parameter, ρ_0 is the equilibrium density of the medium, c_0 is the velocity of sound and b is the coefficient of the viscosity, results in a parameterless model. Moreover, this work considers a modified version of the Westervelt equation given by Model II. As an extension, we construct the Lie point groups for both Westervelt equations under study, also we use modified simple equation method to obtain additional travelling wave solutions.

1.2 Research objectives

In this work, we consider two Westervelt model equations, namely,

- Model I

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 u}{\partial t^3} + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0, \tag{1.9}$$

- Model II

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0. \tag{1.10}$$

1. We perform Lie symmetry analysis on both models to get the determining equations with the help of Mathematica 12.0 [91] software package YaLie [63].
2. We solve the determining equations to get the Lie point symmetries.
3. We show that the symmetries form a Lie algebra via a table of Lie algebras, and construct the table of adjoint representation to obtain the one-dimensional system of sub-algebras.
4. Using the optimal system of sub-algebras, we perform symmetry reductions, that is, reduce our models from partial differential equations to ordinary differential equations.
5. Construct one parameter Lie groups and demonstrate invariance under the one-parameter group of transformations.
6. Generate a family of exact closed form invariant solutions.

1.3 Thesis outline

The dissertation is structured as follows, in Chapter 2, the mathematical preliminaries are presented, that is, the definitions of some of the tools that are to be utilised throughout this work. In Chapter 3, Lie symmetry method is applied to the first model to get the Lie point symmetries. The tables of commutators and adjoint representation for this model are constructed, followed by the one-dimensional optimal system of sub-algebras. Moreover, symmetry reductions are performed to get the invariant sub-models associated with our model. Then, the one-parameter Lie groups associated with our model are constructed and it is demonstrated that the group transformations leave our model invariant. After which, we proceed to find solutions to the invariant sub-models followed by presentations and discussions of results. Chapter 4 is dedicated to the application of Lie symmetry analysis on the second model. We follow the same procedure as we did for the first model. Lastly, in Chapter 5, we give the concluding remarks.

Chapter 2

Mathematical preliminaries

In this chapter, we provide definitions, theorems and special functions used in constructing solutions. This is to construct a firm mathematical background of the ideas that birthed this work. We lay down the theory of Lie symmetry analysis as applied to differential equations by thoroughly exploring some of the key aspects of Lie's method regarding solving differential equations. For an in-depth exploration of the theory of Lie symmetry analysis and its application in solving differential equations, the texts [4, 19, 46, 70] can be consulted.

2.1 Airy functions

These are special functions attributed to an English mathematician by the name of George Bidel Airy, born on 27th July 1801 [1]. It was his work in optics, particularly the calculation of light intensity in the neighbourhood of a caustic that led him to introduce a function

$$W(m) = \int_0^\infty \cos\left[\frac{\pi}{2}(w^2 - mw)\right] dw,$$

which is now known as the Airy function. It is the solution of the differential equation,

$$W'' = \frac{-\pi^2}{12} mW.$$

A new notation adopted by Jefferys in 1928 for the Airy function is

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt,$$

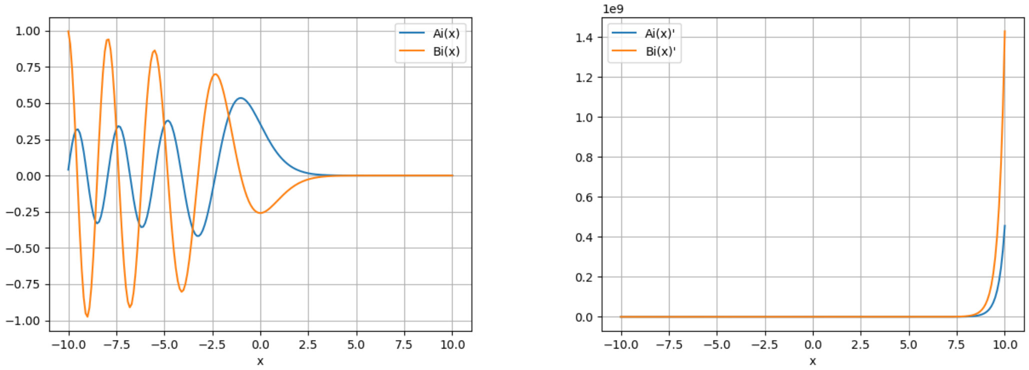
which is the solution of the homogeneous differential equation

$$y''(x) - xy(x) = 0.$$

The Airy function of the second kind is denoted $Bi(x)$, defined as the solution with the same amplitude of oscillation as $Ai(x)$, as $x \to -\infty$, which differs in phase by 2π,

$$Bi(x) = \frac{1}{\pi}\int_0^\infty \left[\exp\left(\frac{-t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt.$$

Below, we provide graphs of the Airy functions and their derivatives, respectively.



(a) Graph of Airy functions

(b) Graph of derivatives of Airy functions

Figure 2.1: In fig. 2.1a, we have $Ai(x)$ and $Bi(x)$ for $-10 \leq x \leq 10$. In fig. 2.1b, we have $Ai'(x)$ and $Bi'(x)$ for $-10 \leq x \leq 10$.

Airy functions have applications in quantum mechanics as solutions to the time-independent Schrödinger equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field [57]. In optics, a non-symmetric optical beam with an electric field profile described by the Airy function has unique characteristics; its peak intensity moves towards one side instead of following a straight path like symmetric beams [26, 98]. However, this causes the lower intensity part of the beam to spread in the opposite direction. Despite this asymmetry, the overall momentum of the beam remains constant (refer to [68]).

2.2 Modified simple equation method

The modified simple equation method is a very efficient method of constructing exact periodic and soliton solutions of some non-linear partial differential equations. This method is based on the assumption that the exact solutions can be represented as polynomials in $\frac{F'}{F}$ such that $F = F(\xi)$, $\xi = x - \alpha t$, is a function to be determined. The main idea is that, F is not pre-defined or not a solution of some pre-defined ordinary differential equation, thus, new solutions maybe found by this method. A thorough outline concerning this method is found in references [8, 47, 60, 96]. The method is summarized in the following steps;

Step 1.

Consider a general form of a non-linear partial differential equation

$$E(u_x, u_t, u_{xx}, u_{xt}, ...) = 0. \tag{2.1}$$

Assume a solution is given by $u(x,t) = y(\xi)$, where $\xi = x - \alpha t$. This transformation changes eq. (2.1) into an ordinary equation given as follows

$$Q(y, y, y'', ...) = 0, \tag{2.2}$$

where Q is a polynomial in the variable y and its derivatives.

Step 2.

Suppose that eq. (2.2) admits the following power series solution

$$y(\xi) = \sum_{i=0}^{M} A_i \left(\frac{F'}{F}\right)^i, \tag{2.3}$$

where A_i are arbitrary constants to be determined such that $A_M \neq 0$, while $F(\xi)$ is an unknown function to be determined later.

Step 3.

Determine the integer M by the principle of homogeneous balance, that is, balancing the highest order derivatives and the non-linear terms.

Step 4.

Substitute eq. (2.3) into eq. (2.2) to get a polynomial in $\frac{F'}{F}$ and its derivatives. In the polynomial, we equate to zero all coefficients of $F(\xi)^i$, $(i = 0, -1, -2, -3, ...)$, which results in a system of equations which can be solved to find A_i and $F(\xi)$, thus, the exact solution of eq. (2.1).

2.3 Group theory

We start the theory of Lie symmetry analysis by briefly looking at the mathematical definition of groups and their properties. For a thorough introduction to group theory, refer to [64].

Definition 2.3.1 *A binary operation $*$ on a set S is a function mapping $S \times S$ into S.*

Definition 2.3.2 *A group $\langle G, * \rangle$ is a set G, closed under a binary operation $*$, such that the following axioms are satisfied,*

1. *Associativity:*

$$(a * b) * c = a * (b * c), \quad \forall a, b, c \in G.$$

2. *Identity:*

   There exists an element $e \in G$ such that

$$e * a = a * e = a, \quad \forall a \in G.$$

3. *Inverse:*

   Corresponding to each $a \in G$, there is an element $a' \in G$ such that

$$a * a' = a' * a = e.$$

4. *Closure:*

   For all $a, b \in G$, if

$$a * b = c,$$

   then $c \in G$.

Definition 2.3.3 *A group G is abelian if its binary operation is commutative, i.e,*

$$a * b = b * a, \quad \forall a, b \in G$$

Definition 2.3.4 *If a subset H of a group G is closed under the binary operation of the group G and if H with the induced operation from G is itself a group, then H is a subgroup.*

2.4 Lie group properties and definitions

The definitions, theorems, corollaries and notations provided in this section are based on [62] and the references therein.

Definition 2.4.1 *A k^{th}-order ($k \geq 1$) system E of s differential equations is defined by*

$$E^{\sigma}(x, u_{(1)}, ..., u_{(k)}) = 0, \quad \sigma = 1, ..., s, \tag{2.4}$$

where $u \equiv (u^1, u^2, .., u^m)$ is the dependent vector, $x \equiv (x^1, x^2, ..., x^n)$ is the independent vector and $u_{(1)}, u_{(2)}, ..., u_{(k)}$ are respectively the collection of all first, second, up to k^{th}-order partial derivatives.

In expanded form

$$u_{(1)} = \{u_i^{\alpha}\}, \quad u_{(2)} = \{u_{ij}^{\alpha}\}, ..., u_{(k)} = \{u_{i_1,...,i_k}^{\alpha}\}$$

where $\alpha = 1, ..., m; j, i_1, ..., i_k = 1, ..., n.$

Definition 2.4.2 *A symmetry transformation of the system in eq.* (2.4) *is an invertible transformation of the dependent variable* u *and independent variables* x*, namely,*

$$x^i = f^i(x, u, a), \quad u^\alpha = \phi^\alpha(x, u, a), \quad i = 1, ..., m, \tag{2.5}$$

that leaves the system in eq. (2.4) *invariant in the new variables* $\bar{x}$ *and* $\bar{u}$*, i.e,*

$$\bar{E}^\sigma(\bar{x}, \bar{u}, \bar{u}_{(1)}, ..., \bar{u}_{(k)}) = 0, \quad \sigma = 1, .., m. \tag{2.6}$$

whenever eq. (2.4) *is satisfied.*

Here, the functions f^i *and* ϕ^α *are differentiable functions and* $a \in \mathbb{R}$ *is a parameter which continuously takes the values in a neighbourhood* $\mathcal{D} \subset \mathcal{R}$ *of* $a = 0$*.*

Definition 2.4.3 *A set* G *of transformations eq.* (2.5)*, namely,*

$$T_a : x^i = f^i(x, u, a),\ u^\alpha = \phi^\alpha(x, u, a), \quad i = 1, ..., n; \alpha = 1, .., m \tag{2.7}$$

is a continuous one-parameter local Lie group of transformations in $\mathbb{R}^{m+n}$ *provided the group properties of closure, identity and inverse hold as follows,*

(a) Closure: If $T_a, T_b \in G$ *and* $a, b \in \mathcal{D}' \subset \mathcal{D}$*, then*

$$T_a T_b = T_c \in G, \quad c = \varphi(a, b) \in \mathcal{D}, \tag{2.8}$$

where $\varphi(a, b)$ *is called the composition law of the group* G*.*

(b) Identity: There exists $T_0 \in G$ *such that*

$$T_0 T_a = T_a T_0 = T_a, \tag{2.9}$$

for any $a \in \mathcal{D}' \subset \mathcal{D}$*.* T_0 *is known as the identity of the group.*

(c) Inverses: There exists $T_a^{-1} = T_{a^{-1}} \in G, a^{-1} \in \mathcal{D}$ *such that*

$$T_a^{-1} T_a = T_a T_a^{-1} = T_0, \tag{2.10}$$

for any $T_a \in G, a \in \mathcal{D}' \subset \mathcal{D}$.

Associativity property follows from eq. (2.8).

Definition 2.4.4 *A group parameter* a *is canonical if the group composition law is additive, i.e,* $\varphi(a, b) = a + b$.

The following theorem shows how a canonical parameter can be constructed.

Theorem 2.4.1 *[44, 69] For any composition law* $\varphi(a, b)$*, there exists a (unique) canonical parameter* $\tilde{a}$ *defined by*

$$\tilde{a} = \int_0^a \frac{d(a')}{\chi(a')},$$

where,

$$\chi(a') = \left.\frac{\partial \varphi(a', b)}{\partial b}\right|_{b=0}.$$

Corollary 2.4.1 *[44, 69] Any one-parameter (local) canonical group is abelian.*

Proof:

Let $T_{a_1}, T_{a_2} \in G$ such that G is any one-parameter (local) canonical group. By Definition 2.4.4, this implies that group composition law is additive, i.e., $\varphi(a_1, a_2) = a_1 + a_2$. Now consider the following,

$$\begin{aligned} \varphi(a_1, a_2) &= a_1 + a_2 \\ &= a_2 + a_1 \\ &= \varphi(a_2, a_1). \end{aligned}$$

Since $\varphi(a_1, a_2) = \varphi(a_2, a_1)$, this implies that $T_{a_1}T_{a_2} = T_{a_2}T_{a_1}$. Therefore we conclude that if G is any one-parameter canonical group then indeed it is abelian $\square$

2.4.1 Infinitesimal generator and Lie's equations

One-parameter group elements can be constructed from their first-order approximations according to Lie's theory. The Taylor expansion of one-parameter group transformations T_a about $a = 0$, with the initial conditions, $f^i|_{a=0} = x^i$ and $\phi^\alpha|_{a=0} = u^\alpha$ is

$$\tilde{x}^i = f^i(x, u, a) \approx f^i(x, u, 0) + a\frac{\partial f^i}{\partial a}\bigg|_{a=0} + \frac{a^2}{2!}\frac{\partial^2 f^i}{\partial a^2}\bigg|_{a=0} + ...$$

$$\tilde{u}^\alpha = \phi^\alpha(x, u, a) \approx \phi^\alpha(x, u, 0) + a\frac{\partial \phi^\alpha}{\partial a}\bigg|_{a=0} + \frac{a^2}{2!}\frac{\partial^2 \phi^\alpha}{\partial a^2}\bigg|_{a=0} + ...$$

Thus,

$$\tilde{x}^i \approx a\xi^i(x, u) + O(a^2), \quad \tilde{u}^\alpha \approx u^\alpha + a\eta(x, u) + O(a^2), \tag{2.11}$$

where,

$$\xi^i(x, u) = \frac{\partial f^i(x, u, a)}{\partial a}\bigg|_{a=0}, \quad \eta^\alpha(x, u) = \frac{\partial \phi^\alpha(x, u, a)}{\partial a}\bigg|_{a=0}. \tag{2.12}$$

The functions in eq. (2.11) are called infinitesimal transformations, they are obtained by the first-order Taylor expansion. The vector (ξ, η) is the tangent vector at (x, u) to the curve $(\bar{x}, \bar{u})$ parameterized by a.

The infinitesimals in eq. (2.11) can be written as,

$$\bar{x}^i \approx (1 + aX)x^i, \quad \bar{u}^\alpha \approx (1 + aX)u^\alpha, \tag{2.13}$$

where

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{2.14}$$

The differential operator in eq. (2.14) is called the infinitesimal generator (operator) or vector field of the group G. It is also known as the symmetry generator (operator).

Theorem 2.4.2 *[44, 69] For any given infinitesimal transformations or an infinitesimal generator, the corresponding one-parameter group G is obtained by solution of Lie's equations,*

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x}, \bar{u}), \quad \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x}, \bar{u}), \tag{2.15}$$

subject to the initial conditions

$$\bar{x}^i|_{a=0} = x^i, \quad \bar{u}^\alpha|_{a=0} = u^\alpha.$$

2.4.2 Invariants

Definition 2.4.5 *A point $(x, u) \in \mathbb{R}^{n+m}$ is an invariant point if it remains unchanged by every transformation of a group G. i.e,*

$$(\bar{x}, \bar{u}) = (x, u)\,, \quad \forall a \in \mathcal{D}' \subset \mathcal{D}.$$

Theorem 2.4.3 *[44, 69] A point $(x, u) \in \mathbb{R}^{n+m}$ is an invariant point of a group G with generator*

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}.$$

if and only if

$$\xi^i(x, u) = \eta^\alpha(x, u) = 0.$$

Definition 2.4.6 *A function $\mathbb{F}(x, u)$ is an invariant of a group G if and only if*

$$\mathbb{F}(\bar{x}, \bar{u}) = \mathbb{F}(x, u), \quad \forall x, u, a \in \mathcal{D}' \subset \mathcal{D}.$$

Definition 2.4.7 *A function $\mathbb{F}(x,u)$ is an invariant of a group G with generator X if and only if*

$$X\left(\mathbb{F}\right)=0. \tag{2.16}$$

The characteristic system for eq. (2.16) is given by

$$\frac{dx^1}{d\xi^1(x,u)}=...=\frac{dx^n}{d\xi^n(x,u)}=\frac{du^1}{d\eta^1(x,u)},...=\frac{du^m}{d\eta^m(x,u)}. \tag{2.17}$$

Thus, an arbitrary invariant $\mathbb{F}(x,u)$ of the group G is

$$\mathbb{F}=\Lambda(I_1(x,u),...,I_{m+n-1}(x,u)),$$

where $I_1(x,u),...,I_{m+n-1}(x,u)$ is called the basis of invariants of G (i.e, group G has exactly $m+n-1$ functionally independent invariants). The basis is not unique. One can take, as basic invariants, the left hand side of $m+n-1$ first integrals

$$I_i(x,u),...,I_{m+n-1}(x,u)=c_{m+n-1}.$$

2.4.3 Extended groups

The transformations in eq. (2.5) form a symmetry group G of the system in eq. (2.4) if its invariant form in eq. (2.6) is satisfied whenever eq. (2.4) holds. The transformed derivatives in eq. (2.6) are obtained by employing the chain rule, $D_i=D_i(f^j)\bar{D}_j$, where,

$$D_j=\frac{\partial}{\partial x^j}+u^\alpha_j\frac{\partial}{\partial u^\alpha}+u^\alpha_{jk}\frac{\partial}{\partial u^\alpha_k}+...\quad;u^\alpha_j=D_j(u^\alpha),\ \ u^\alpha_{jk}=D_j(u^\alpha_k),$$

is the total differential operator with respect to x^i, as is $\bar{D}_i$ for the transformed variables. Applying $D_i=D_i(f^j)\bar{D}_j$ on $\bar{u}^\alpha$ from eq. (2.5), we arrive at

$$
\begin{aligned}
D_i(\phi^\alpha) &= D_i(f^j)\bar{D}_j(\bar{u}^\alpha)\\
&= D_i(f^j)\bar{u}^\alpha_j
\end{aligned}
$$

Expanding the above, we get

$$\frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta} = \left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right)\bar{u}_j^\alpha, \tag{2.18}$$

and solving eq. (2.18) for $\bar{u}_j^\alpha$, we obtain,

$$\bar{u}_j^\alpha = \left(\frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}\right) \Big/ \left(\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta}\right), \tag{2.19}$$

thus, $\bar{u}_j^\alpha = \varphi_i^\alpha(x, u, u_{(i)}, a)$ for a small a subject to $\varphi_i^\alpha|_{a=0} = u_i^\alpha$. The transformations in eq. (2.5) together with the transformation $\bar{u}_{(1)} = \varphi(x.u, u_{(1)}, a)$ form a one-parameter group, $G^{[1]}$, which is the first prolonged group acting on the space $(x, u, u_{(1)})$. In a similar fashion, high-order prolonged (extended) groups, $G^{[2]}$ up to $G^{[k]}$ can be obtained.

2.4.4 Prolongation formulas

Definition 2.4.8 *The extended infinitesimal generator $X^{[k]}$ of the prolonged group $G^{[k]}$ on the space $(t, x, u, u_{(1)}, u_{(2)}, ..., u_{(k)})$ is called the k-th prolongation of X and denoted by*

$$X^{[k]} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_i \frac{\partial}{\partial u_{(1)}} + ... + \zeta_{i_1 \ldots i_k} \frac{\partial}{\partial u_{(i_1 \ldots i_k)}}. \tag{2.20}$$

The coefficients ζ's are determined recursively by the prolongation formulae,

$$\zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ji}^\alpha,$$

$$\zeta_{ij}^\alpha = D_i D_j(W^\alpha) + \xi^k u_{kj}^\alpha,$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\zeta_{i_1 \ldots i_k}^\alpha = D_{i_1} \ldots D_{i_k}(W^\alpha) + \xi^j u_{j i_1 \ldots i_k}^\alpha,$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha,$$

is called Lie characteristic function.

2.4.5 Lie point symmetries and Lie's algorithm

Theorem 2.4.4 *[44,69] Let G be a group of infinitesimal transformations admitted by a system E. Then G consists of symmetries of the system E if and only if*

$$X^{[k]}(E^\sigma(x, u, u_{(1)}, .., u_{(k)})) = 0, \quad \sigma = 1, ..., s$$

whenever eq. (2.4) is satisfied for every group generator X of G.

Lie's algorithm [62].

Below we give a layout of the steps involved in the execution of the procedure for calculating symmetries of the system E in eq. (2.4).

1. Write E such that all the terms are on the left hand side.

2. Write the generator of symmetry

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u)\frac{\partial}{\partial u^\alpha}.$$

3. Prolong the symmetry generator X to the order which is the same as that of E, i.e,

$$X^{[k]} = X + \zeta_i^\alpha(x, u, u_{(1)})\frac{\partial}{\partial u_i^\alpha} + ... + \zeta_{i_1 ... i_k}^\alpha(x, u, ..., u_{(k)})\frac{\partial}{\partial u_{i_1 ... i_k}^\alpha},$$

   where the variables are given by the prolongation formulas.

4. Apply the the prolonged generator $X^{[k]}$ on E evaluate on the surface in eq. (2.4) yielding the symmetry conditions

$$X^{[k]}(E^\sigma(x, u, u_{(1)}, ..., u_{(k)}))|_{(2.4)} = 0, \quad \sigma = 1, ..., s. \tag{2.21}$$

5. Substitute ζ_i^α upon expansion of the symmetry condition and replace the derivatives that are to be eliminated.

6. Separate the expanded expression with respect to the derivatives of the dependent variables and their powers resulting in an over-determined system of linear homogeneous PDEs in terms of ξ^i and η^α.

7. Solve the over-determined system for the infinitesimals ξ^i and η^α to obtain symmetries of E.

8. Construct one-parameter groups using Theorem 2.4.2.

2.4.6 Lie algebras and adjoint representation

Definition 2.4.9 *A Lie algebra is formed by a vector space $\mathbb{L}$ over a field, $\mathcal{F}$, say, together with a binary operation, $[\cdot, \cdot]$ called a Lie bracket, commonly known as a commutator, defined on $\mathbb{L}$ such that the following properties hold,*

- *Suppose that X_a and X_b are $\in \mathbb{L}$ and α,$\beta \in \mathbb{R}$ then, the Lie algebra is a vector space spanned by the basis set of infinitesimal generators such that,*

$$X = \alpha X_a + \beta X_b, \tag{2.22}$$

where X $\in \mathbb{L}$.

- *The commutator is anti-symmetric*

$$[X_a, X_b] = -[X_b, X_a]. \tag{2.23}$$

- *The group operators satisfy the Jacobi identity*

$$[[X_a, X_b], X_c] + [[X_c, X_a], X_b] + [[X_b, X_c], X_a] = 0. \tag{2.24}$$

- *The commutator is bilinear*

$$[[\alpha X_a + \beta X_b], X_c] = \alpha[X_a, X_c] + \beta[X_b, X_c]. \tag{2.25}$$

Definition 2.4.10 *Consider a Lie Algebra* $\mathbb{L}$*, if the vector space* $\mathbb{L}$ *is finite-dimensional, its dimension is that of the Lie algebra, that is, the finite dimensional algebra of dimension* r *is denoted by* $\mathbb{L}_r$*. For this work the bracket* $[\cdot, \cdot]$ *is defined as follows*

$$[X_a, X_b] = X_a(X_b) - X_b(X_a). \tag{2.26}$$

Definition 2.4.11 *The adjoint representation is given by the formula*

$$Ad(\exp{(\epsilon X_i)})X_j = X_j - \epsilon[X_i, X_j] + \tfrac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \tfrac{\epsilon^3}{3!}[X_i, [X_i, [X_i, X_j]]] + ... \tag{2.27}$$

Definition 2.4.12 *An optimal system of a Lie algebra is a set of* l*-dimensional subalgebras such that every* l*-dimensional subalgebra is equivalent to a unique element of the set under some element of adjoint representation.*

Chapter 3

Lie symmetry analysis of the Westervelt equation: Model I

In this chapter, we consider the Westervelt equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^4}\frac{\partial^3 u}{\partial t^3} + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0, \tag{3.1}$$

which will be referred to as Model I in this work [90]. The dissipation term is a third-order temporal partial derivative, thus eq. (3.1) is suitable for modelling non-Newtonian fluids[1]. We perform Lie symmetry analysis, which involves finding Lie point symmetries, constructing optimal systems, and subsequently generating one-parameter Lie groups and invariant sub-models, which are solved to get the exact solutions associated with this model. Finally, we present and discuss the results.

3.1 Determining equations and infinitesimal generator

The computation of determining equations by hand can be a tedious task, especially when the partial differential equation is of order 2 or more. Therefore, since our model is of order 3, we have employed a computer software package for symbolic computations (YaLie). This package

---

[1] The viscosity of a non-Newtonian fluid is time dependent as such a model that has only temporal derivatives for the dissipation term can best describe the propagation of sound through a Non-Newtonian fluid.

helps in the generation and manipulation of determining equations for the maximal symmetry Lie algebra.

We begin by rewriting our model in eq. (3.1) as follows,

$$u_{ttt} = Bu_{tt} - Au_{xx} - 2C(u_t)^2 - 2Cuu_{tt}, \tag{3.2}$$

where $A = \frac{c_o^4}{\delta}$, $B = \frac{c_0^2}{\delta}$ and $C = \frac{\beta}{\rho\delta}$, such that $\delta \neq 0$, $\beta \neq 0$, $c_0 \neq 0$ and $\rho \neq 0$. The infinitesimal generator of eq. (3.2) is given by,

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{3.3}$$

provided,

$$\mathbf{Pr}^{[3]}X(Au_{xx} - Bu_{tt} + u_{ttt} + 2C(u_t)^2 + 2Cuu_{tt})|_{(3.2)} = 0,$$

where the third prolongation of X is given as follows,

$$\begin{aligned}\mathbf{Pr}^{[3]}X = {} & \xi^1\partial_x + \xi^2\partial_t + \eta\partial_u + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{tt}\partial_{u_{tt}} + \zeta_{tx}\partial_{u_{tx}} + \zeta_{xx}\partial_{u_{xx}} + \zeta_{ttt}\partial_{u_{ttt}} \\ & + \zeta_{ttx}\partial_{u_{ttx}} + \zeta_{txx}\partial_{u_{txx}} + \zeta_{xxx}\partial_{u_{xxx}},\end{aligned}$$

such that,

$$\zeta_t = D_t(\eta) - u_t D_t(\xi^1) - u_x D_t(\xi^2), \tag{3.4a}$$

$$\zeta_x = D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2), \tag{3.4b}$$

$$\zeta_{tt} = D_t(\zeta_t) - u_{tt} D_t(\xi^1) - u_{xt} D_t(\xi^2), \tag{3.4c}$$

$$\zeta_{tx} = D_t(\zeta_x) - u_{tt} D_t(\xi^1) - u_{tx} D_t(\xi^2), \tag{3.4d}$$

$$\zeta_{xx} = D_x(\zeta_x) - u_{tx} D_x(\xi^1) - u_{xx} D_x(\xi^2), \tag{3.4e}$$

$$\zeta_{ttt} = D_t(\zeta_{tt}) - u_{ttx} D_t(\xi^1) - u_{ttt} D_t(\xi^2), \tag{3.4f}$$

$$\zeta_{ttx} = D_x(\zeta_{tt}) - u_{ttx} D_x(\xi^1) - u_{ttt} D_x(\xi^2), \tag{3.4g}$$

$$\zeta_{txx} = D_t(\zeta_{xx}) - u_{xxt} D_t(\xi^1) - u_{xxx} D_t(\xi^2), \tag{3.4h}$$

$$\zeta_{xxx} = D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2). \tag{3.4i}$$

The total differential operators D_t and D_x are,

$$D_t = \partial_t + u_t \partial_u + u_{tt} \partial_{u_t} + u_{ttt} \partial_{u_{tt}} + ...,$$

$$D_x = \partial_x + u_x \partial_u + u_{xx} \partial_{u_x} + u_{xxx} \partial_{u_{xx}} + ...$$

Thus, ξ^1, ξ^2 and η are the infinitesimals to be determined. Therefore the prolonged generator, acting on eq. (3.2) yields,

$$2C\eta u_{tt} + 4C\zeta_t u_t - B\zeta_{tt} + 2C\zeta_{tt} u + A\zeta_{xx} + \zeta_{ttt} = 0. \tag{3.5}$$

Upon substituting the expanded forms of the ζ's and the total differential operators into eq. (3.5) we obtain the following equation

$$
\begin{aligned}
&2C\eta u_{tt} - 3u_{tt}u_{tx}\xi^1_u - 2Au_xu_{xx}\xi^1_u - 3u_{tt}^2\xi^2_u - 2Au_xu_{xt}\xi^2_u + Au_x^2\eta_{uu} - Au_x^3\xi^1_{uu} - u_t^4\xi^2_{uuu} \\
&+ 2Bu_{xt}\xi^1_t - 4Cuu_{xt}\xi^1_t - 3u_{xtt}\xi^1_t - Bu_{tt}\xi^2_t + 2Cuu_{tt}\xi^2_t + 3Au_{xx}\xi^2_t + 3u_{tt}\eta_{tu} \\
&- 3u_{tt}u_x\xi^1_{tu} + u_t^3(4C\xi^2_u + (B - 2Cu)\xi^2_{uu} + \eta_{uuu} - u_x\xi^1_{uuu} - 3\xi^2_{tuu}) + B\eta_{tt} + 2Cu\eta_{tt} \\
&+ Bu_x\xi^1_{tt} - 2Cuu_x\xi^1_{tt} - 3u_{xt}\xi^1_{tt} - 3u_{tt}\xi^2_{tt} + u_t^2(2C\eta_u - B\eta_{uu}2Cu\eta_{uu} - 3u_{tx}\xi^1_{tt} \\
&- 3u_{tt}\xi^2_{tt} + 2B\xi^2_{tu} - 4Cu\xi^2_{tu} + 3\eta_{tuu} + u_x(-2C\xi^1_u + (B - 2Cu)\xi^1_{uu} - 3\xi^1_{tuu}) - 3\xi^2_{ttu}) \\
&+ \eta_{tt} - u_x\xi^1_{ttt} - 2Au_{xx}\xi^1_x - 2Au_{xt}\xi^2_x + 2Au_x\eta_{xu} - 2Au_x^2\xi^1_{xu} + A\eta_{xx} - Au_x\xi^1_{xx} \\
&+ u_t(-3u_{xtt}\xi^1_u - Bu_{tt}\xi^2_u + 2Cuu_{tt}\xi^2_u + 3Au_{xx}\xi^2_u + 3u_{tt}\eta_{uu} - 3u_{tt}u_x\xi^1_{uu} - Au_x^2\xi^2_{uu} \\
&+ 4C\eta_t - 4Cu_x\xi^1_t - 2B\eta_{tu} + 4Cu\eta_{tu} + 2u_{xt}((B - 2Cu)\xi^1_u - 3\xi^1_{tu}) + 2Bu_x\xi^1_{tu} \\
&- 4Cuu_x\xi^1_{tu} - 9u_{tt}\xi^2_{tu} + B\xi^2_{tt} - 2Cu\xi^2_{tt} + 3\eta_{ttu} - 3u_x\xi^1_{ttu} - \xi^2_{ttt} - 2Au_x\xi^2_{xu} - A\xi^2_{xx}) \\
&= 0.
\end{aligned}
\tag{3.6}
$$

Separating eq. (3.6) with respect to the powers of the derivatives of the dependent variable u and their products, the following system of determining equations is obtained,

$$\xi^1_u = 0, \tag{3.7a}$$

$$\xi^1_t = 0, \tag{3.7b}$$

$$\xi^2_u = 0, \tag{3.7c}$$

$$\xi^2_x = 0, \tag{3.7d}$$

$$\eta_{uu} = 0, \tag{3.7e}$$

$$3A\xi^2_t - 2A\xi^1_x = 0, \tag{3.7f}$$

$$2A\eta_{xu} - A\xi^1_{xx} = 0, \tag{3.7g}$$

$$2C\eta_u + 2C\xi^2_t = 0, \tag{3.7h}$$

$$(-B + 2Cu)\eta_{tt} + \eta_{ttt} + A\eta_{xx} = 0, \tag{3.7i}$$

$$4C\eta_t + 2(-B + 2Cu)\eta_{tu} + (B - 2Cu)\xi^2_{tt} + 3\eta_{ttu} - \xi^2_{ttt} = 0, \tag{3.7j}$$

$$2C\eta + (-3B + 6Cu - 2(-B + 2Cu))\xi^2_t + 3\eta_{tu} - 3\xi^2_{tt} = 0. \tag{3.7k}$$

From eqs. (3.7a) to (3.7e), we note that,

$$\xi^1 = \xi^1(x), \tag{3.8}$$

$$\xi^2 = \xi^2(t), \tag{3.9}$$

$$\eta = f(t,x)u + g(t,x). \tag{3.10}$$

Updating our system and taking into consideration eqs. (3.8) to (3.10) above we obtain the following simplified system of equations,

$$2Cf + 2C\xi^2_t = 0, \tag{3.11a}$$

$$-2A\xi^1_x + 3A\xi^2_t = 0, \tag{3.11b}$$

$$-A\xi^1_{xx} + 2Af_x = 0, \tag{3.11c}$$

$$2Cuf + 2Cg - B\xi^2_t + 2Cu\xi^2_t - 3\xi^2_{tt} - 3f_t = 0, \tag{3.11d}$$

$$B\xi^2_{tt} - 2Cu\xi^2_{tt} - \xi^2_{ttt} - 2Bf_t + 8Cuf_t + 4Cg_t + 3f_{tt} = 0, \tag{3.11e}$$

$$-Buf_{tt} + 2Cu^2f_{tt} - Bg_{tt} + 2Cug_{tt} + uf_{ttt} + g_{ttt} + Auf_{xx} + Ag_{xx} = 0. \tag{3.11f}$$

Considering eq. (3.11a), we see that,

$$f(t,x) = -\xi^2_t. \tag{3.12}$$

We further update eq. (3.10) by substituting eq. (3.12) into eq. (3.10) to get the infinitesimals,

$$\xi^1 = \xi^1(x), \tag{3.13}$$

$$\xi^2 = \xi^2(t), \tag{3.14}$$

$$\eta = (-\xi^2_t)u + g(t,x). \tag{3.15}$$

Using equations eqs. (3.13) to (3.15), the following system is obtained,

$$-A\xi^1_{xx} = 0, \tag{3.16a}$$

$$-2A\xi^1_x + 3A\xi^2_t = 0, \tag{3.16b}$$

$$2Cg - B\xi^2_t - 6\xi^2_{tt} = 0, \tag{3.16c}$$

$$3B\xi^2_{tt} - 10Cu\xi^2_{tt} - 4\xi^2_{ttt} + 4Cg_t = 0, \tag{3.16d}$$

$$Bu\xi^2_{ttt} - 2Cu^2\xi^2_{ttt} - u\xi^2_{tttt} - Bg_{tt} + 2Cug_{tt} + g_{ttt} + Ag_{xx} = 0. \tag{3.16e}$$

Differentiating eq. (3.16b) with respect to t, while taking into consideration eq. (3.13), we obtain

$$\xi^2_{tt} = 0, \tag{3.17}$$

therefore,

$$\xi^2_{ttt} = \xi^2_{tttt} = 0. \tag{3.18}$$

Now considering eq. (3.16c), we get

$$g(t,x) = \frac{B}{2C}\xi^2_t. \tag{3.19}$$

Substituting eq. (3.19) into eq. (3.15),

$$\eta = (-\xi^2_t)u + \frac{B}{2C}\xi^2_t. \tag{3.20}$$

Updating η given by eq. (3.20) in conjunction with eq. (3.13) and eq. (3.14) results in the following system of determining equations,

$$-6\xi^2_{tt} = 0, \tag{3.21a}$$

$$-A\xi^1_{xx} = 0, \tag{3.21b}$$

$$-2A\xi^1_x + 3A\xi^2_t = 0, \tag{3.21c}$$

$$5B\xi^2_{tt} - 10Cu\xi^2_{tt} - 4\xi^2_{ttt} = 0, \tag{3.21d}$$

$$\frac{-B^2}{2C}\xi^2_{ttt} + 2Bu\xi^2_{ttt} - 2Cu^2\xi^2_{ttt} + \frac{B}{2C}\xi^2_{tttt} - u\xi^2_{tttt} = 0. \tag{3.21e}$$

We consider eq. (3.21b), which implies that,

$$\xi^1_x = k_1. \tag{3.22}$$

Upon substituting eq. (3.22) into eq. (3.21c), we obtain

$$\xi^2_t = \frac{2}{3}k_1. \tag{3.23}$$

We integrate eq. (3.22) with respect to x and eq. (3.23) with respect to t to get the explicit forms of ξ^1 and ξ^2, therefore the infinitesimals are

$$\xi^1(x) = xk_1 + k_1, \tag{3.24a}$$

$$\xi^2(t) = \frac{2}{3}tk_1 + k_3, \tag{3.24b}$$

$$\eta(u) = \left(\frac{B}{3C} - \frac{2u}{3}\right)k_1. \tag{3.24c}$$

The symmetry generator in eq. (3.3) is given by

$$X = (xk_1 + k_2)\,\partial_x + \left(\frac{2}{3}tk_1 + k_3\right)\partial_t + \left(\frac{B}{3C} - \frac{2}{3}u\right)k_1\partial_u. \tag{3.25}$$

3.2 Lie point symmetries

The Lie point symmetries for Model I are found by considering the symmetry generator in eq. (3.25) where the number of constants, i.e., k_i's, corresponds to the number of Lie point symmetries. Model I admits a three dimensional Lie algebra, i.e, $\mathbb{L}_3$, with the following basis,

$$X_1 = x\partial_x + \frac{2}{3}t\partial_t + \left(\frac{B}{3C} - \frac{2}{3}u\right)\partial_u \approx 3Cx\partial_x + 2Ct\partial_t + (B - 2Cu)\partial_u, \tag{3.26a}$$

$$X_2 = \partial_x, \tag{3.26b}$$

$$X_3 = \partial_t. \tag{3.26c}$$

3.3 Optimal system of sub-algebras

In this section, we construct a one-dimensional optimal system for the Lie algebra with basis eqs. (3.26a) to (3.26c). The symmetry Lie algebra is utilised for the symmetry reductions and the construction of invariant solutions. Some linear combination of the Lie point symmetries can be used for symmetry reductions to obtain invariant solutions.

It can happen that there are many Lie point symmetries, as a result there will be many linear combinations. However, some symmetries are connected with each other by a transformation from the symmetry group, meaning their invariant solutions are also connected by the same transformation. It is therefore necessary to put into one class such symmetries, a systematic way of classifying such combinations is called obtaining an optimal system of sub-algebras. This corresponds to non-unique linear combinations, which represent all possible linear combinations.

There are three generally used methods of obtaining an optimal system. The first is where the Lie algebra is overly simplified by performing various adjoint transformations [62], the second method entails finding the sub-algebras of the given Lie algebra and grouping them into conjugate classes [71]. Conjugacy in each case is considered under the group of inner automorphism, i.e., the Lie group found by exponentiation of the adjoint representation of the Lie algebra under consideration. Lastly, a global matrix for the adjoint transformation is used to construct an optimal system [41]. In this work, we use the first method because it bypasses the laborious process of calculating the matrix of global adjoints or creating a group of inner automorphisms.

Using Definitions 2.4.10 and 2.4.11, respectively, we construct table of commutators and a table of adjoint representations corresponding to eqs. (3.26a) to (3.26c) are given below Tables 3.1 and 3.2.

$\nearrow [X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-3CX_2$	$-2CX_3$
X_2	3CX_2	0	0
X_3	$2CX_3$	0	0

Table 3.1: Lie Algebras

$Ad(e^{\epsilon X_i})X_j$	X_1	X_2	X_3
X_1	X_1	$e^{3C\epsilon}X_2$	$e^{2C\epsilon}X_3$
X_2	$X_1 - 3C\epsilon X_2$	X_2	X_3
X_3	$X_1 - 2C\epsilon X_3$	X_2	X_3

Table 3.2: Adjoint representation

To construct the one-dimensional optimal sub-algebra for the symmetry group $\langle X_1, X_2, X_3 \rangle$, we consider the general symmetry operator

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{3.27}$$

for some arbitrary constants $a_i, (i = 1, 2, 3)$. We investigate if the effect of applying the adjoint action unto eq. (3.27) will reduce it to a simpler form. We begin by assuming $a_1 \neq 0$ and rescale a_1, such that $a_1 = 1$, thus,

$$X = a_2X_2 + a_3X_3 + X_1. \tag{3.28}$$

We act on eq. (3.28) by $Ad(e^{\frac{a_2}{3C}X_2})$ to eliminate a_2X_2 as follows,

$$\begin{aligned} X' &= Ad(e^{\frac{a_2}{3C}X_2})X \\ &= Ad(e^{\frac{a_2}{3C}X_2})(a_2X_2 + a_3X_3 + X_1) \\ &= a_2Ad(e^{\frac{a_2}{3C}X_2})X_2 + a_3Ad(e^{\frac{a_2}{3C}X_2})X_3 + Ad(e^{\frac{a_2}{3C}X_2})X_1 \\ &= a_2X_2 + a_3X_3 + X_1 - 3C(\frac{a_2}{3C})X_2 \\ &= a_3X_3 + X_1. \end{aligned} \tag{3.29}$$

Again, we act on eq. (3.29) by $Ad(e^{\frac{a_3}{2C}X_3})$ to eliminate a_3X_3 in the following manner,

$$\begin{aligned} X'' &= Ad(e^{\frac{a_3}{2C}X_3})X' \\ &= Ad(e^{\frac{a_3}{2C}X_3})(a_3X_3 + X_1) \\ &= a_3X_3 + X_1 - 2C(\frac{a_3}{2C})X_3 \\ &= X_1. \end{aligned} \tag{3.30}$$

Therefore, X is equivalent to X_1 under the adjoint representation, meaning, any one-dimensional sub-algebra generated by X, such that $a_1 \neq 0$, is equivalent to a sub-algebra spanned by X_1. Now, suppose $a_1 = 0$ and $a_2 \neq 0$. We rescale a_2 such that, $a_2 = 1$, to obtain

$$X = X_2 + a_3X_3. \tag{3.31}$$

Acting on eq. (3.31) with $Ad(e^{\epsilon X_1})$, i.e., group generated by X_1 we get,

$$X' = e^{3C\epsilon}X_2 + a_3e^{2C\epsilon}X_3, \tag{3.32}$$

which is a scalar multiple of eq. (3.31). Depending on the sign of a_3, we can make the coefficient of X_3 to be $+1, -1$. Hence, any one-dimensional sub-algebra generated by X, where $a_1 = 0$ and $a_2 \neq 0$, is the same as a sub-algebra spanned by $X_2 + \Gamma X_3;\ \ \Gamma = \pm 1$. Thus, for $a_3 > 0 :$ $X = X_2 + X_3$, $a_3 < 0 : X = X_2 - X_3$.

Also, suppose that $a_1 = a_3 = 0$ and $a_2 \neq 0$, such that $a_2 = 1$, we obtain

$$X = X_2. \tag{3.33}$$

Hence, any one-dimensional sub-algebra generated by X, such that $a_1 = a_3 = 0$ and $a_2 \neq 0$, is equivalent to a sub-algebra spanned by X_2. Finally, $a_1 = a_2 = 0 : X = X_3$, meaning any one-dimensional sub-algebra generated by X, such that $a_3 \neq 0$ and $a_1 = a_2 = 0$, is equivalent to a sub-algebra spanned by X_3. Thus, the one-dimensional optimal system of sub-algebras is spanned by set

$$\{X_1, X_2, X_3, X_2 + X_3, X_2 - X_3\}\,. \tag{3.34}$$

3.4 Symmetry reductions and invariant sub-models

In this section we utilise the one-dimensional optimal system of sub-algebras found in Section 3.3 to reduce the parent model into invariant sub-models. We employ the symmetries together with their respective combinations to transform the governing model from a partial differential equation into an ordinary differential equation. This is done by employing Lagrange's method of characteristic curves [61].

It is worth noting that any linear combination between the time and space translations, i.e., ∂_t and ∂_x, will result in the most general invariant sub-model which obeys the traveling wave solution, that is, $u(x,t) = g(x - \alpha t)$.

3.4.1 Invariance under X_1.

The characteristic equation is given by,

$$\frac{dx}{3Cx} = \frac{dt}{2Ct} = \frac{du}{B - 2Cu}. \tag{3.35}$$

Considering the first and second ratios from the characteristic equation, we get,

$$2\frac{dx}{x} = 3\frac{dt}{t}. \tag{3.36}$$

Hence eq. (3.36) implies $\frac{x^2}{t^3} = l_1$, where l_1 is the first invariant. Next, take the second and last ratios of eq. (3.35),

$$\frac{dt}{2Ct} = \frac{du}{B - 2Cu}. \tag{3.37}$$

We can rewrite eq. (3.37) as a first-order linear differential equation as follows

$$\frac{du}{dt} + \frac{u}{t} - \frac{B}{2Ct} = 0, \tag{3.38}$$

upon solving eq. (3.38), we get the second invariant, $l_2 = ut - \frac{Bt}{2C}$[2]. The invariance condition $\mathcal{F}(l_1) = l_2$, gives the general solution,

$$u(x,t) = \frac{\mathcal{F}}{t} + \frac{B}{2C}. \tag{3.39}$$

Substituting eq. (3.39) into eq. (3.2), we get the following ordinary differential equation

$$\begin{aligned}&-6\mathcal{F} + 6C(\mathcal{F})^2 + 2A\mathcal{F}' - 144l_1\mathcal{F}' + 48Cl_1\mathcal{F}\mathcal{F}' + 18C(l_1\mathcal{F}')^2 + 4Al_1\mathcal{F}'' \\ &-135(l_1)^2\mathcal{F}'' + 18C(l_1)^2\mathcal{F}\mathcal{F}'' - 27(l_1)^3\mathcal{F}''' = 0,\end{aligned} \tag{3.40}$$

where $\mathcal{F}' = \frac{d\mathcal{F}}{dl_1}$, $\mathcal{F}'' = \frac{d^2\mathcal{F}}{dl_1^2}$ and $\mathcal{F}''' = \frac{d^3\mathcal{F}}{dl_1^3}$.

3.4.2 Invariance under X_2.

The associated characteristic equation is,

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \tag{3.41}$$

which results in the following invariants, $t = a_1$, and $u = a_2$. The invariant solution is given by $a_2 = \mathcal{W}(a_1)$, i.e., $u(x,t) = \mathcal{W}(t)$. Substituting the invariant solution into eq. (3.2) reduces the model into the following ordinary differential equation,

$$\mathcal{W}''' - B\mathcal{W}'' + 2C(\mathcal{W}')^2 + 2C\mathcal{W}\mathcal{W}'' = 0, \tag{3.42}$$

where $\mathcal{W}' = \frac{d\mathcal{W}}{dt}$, $\mathcal{W}'' = \frac{d^2\mathcal{W}}{dt^2}$, $\mathcal{W}''' = \frac{d^3\mathcal{W}}{dt^3}$.

---

[2] Since eq. (3.38) contains no derivative with respect to the x variable, we can regard this variable as a parameter.

3.4.3 Invariance under X_3.

The corresponding characteristic equation is,

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}, \tag{3.43}$$

solving yields the following invariants, $x = p_1$, and $u = p_2$. The invariant solution is given by $p_2 = \mathcal{D}(p_1)$, i.e., $u(x,t) = \mathcal{D}(x)$. Substituting the invariant solution into eq. (3.2) reduces the model into the ordinary differential equation,

$$A\mathcal{D}'' = 0, \tag{3.44}$$

which implies,

$$\mathcal{D}'' = 0, \tag{3.45}$$

where $\mathcal{D}'' = \frac{d^2\mathcal{D}}{dx^2}$. Therefore, eq. (3.45) has a trivial (linear) solution independent of time given below

$$u(x,t) = z_1 x + z_2, \tag{3.46}$$

such that z_i, $(i = 1, 2)$, are the constants of integration.

3.4.4 Invariance under $X_3 + \alpha X_2$.

The parameter α, represents the wave-speed. From the described symmetry, we get the characteristic equation,

$$\frac{dx}{\alpha} = \frac{dt}{1} = \frac{du}{0}. \tag{3.47}$$

Considering the last ratio of eq. (3.47), we get that $u = J_1$ as the first invariant, then taking into account the first and second ratios of eq. (3.47), we get $x - \alpha t = J_2$ as the second invariant. The invariance condition $\mathcal{G}(J_2) = J_1$ gives a general solution of the form,

$$u(x,t) = \mathcal{G}(J_2). \tag{3.48}$$

Substituting eq. (3.48) into eq. (3.2) results in the ordinary differential equation,

$$\mathcal{G}''' + \left(\frac{B}{\alpha} - \frac{A}{\alpha^3}\right)\mathcal{G}'' - \frac{2C}{\alpha}(\mathcal{G}')^2 - \frac{2C}{\alpha}\mathcal{G}\mathcal{G}'' = 0, \tag{3.49}$$

where $\mathcal{G}' = \frac{d\mathcal{G}}{dJ_2}$, $\mathcal{G}'' = \frac{d^2\mathcal{G}}{dJ_2^2}$ and $\mathcal{G}''' = \frac{d^3\mathcal{G}}{dJ_2^3}$.

3.5 Results and Discussions

This section presents the results for Model I. According to Lie's algorithm in Subsection 2.4.5, one of the aims of this method is to construct one-parameter Lie groups which are used to map a solution of the same differential equation unto another solution via point group transformations. We begin with the computation of the one-parameter Lie groups for our model and demonstrate that any one-parameter Lie group picked at random leaves the partial differential equation invariant.

Furthermore, we present exact solutions of the invariant sub-models that were obtained in Section 3.4 as a result of the symmetry reductions. The solutions found in this section are generated by a computer algebra software (Wolfram Mathematica 12.0.), while some solutions are found using the modified simple equation method explained in Section 2.2. There are, however invariant sub-models that are unsolvable either from being too non-linear, any attempts to such solutions are not presented. A few of the exact solutions are expressed in terms of the Airy functions and their derivatives, respectively, a brief introduction on these functions is provided in Section 2.1.

3.5.1 One-parameter Lie groups

According to Theorem 2.4.2 one parameter Lie groups are obtained by solving Lie's equations. Now, taking into consideration the symmetries found in eqs. (3.26a) to (3.26c), we solve the resulting ordinary differential equations to construct the one parameter Lie groups for some parameter vector $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$.

Consider the first Lie point symmetry, we have,

$$X_1 = 3Cx\partial_x + 2Ct\partial_t + (B - 2Cu)\partial_u.$$

The corresponding Lie's equations are

$$\xi^1(\bar{x}, \bar{t}, \bar{u}) = \frac{d\bar{x}}{d\epsilon_1}, \quad \xi^2(\bar{x}, \bar{t}, \bar{u}) = \frac{d\bar{t}}{d\epsilon_1}, \quad \eta(\bar{x}, \bar{t}, \bar{u}) = \frac{d\bar{u}}{d\epsilon_1}, \tag{3.50}$$

where,

$$\begin{cases} \xi^1(\bar{x}, \bar{t}, \bar{u}) & = 3C\bar{x}, \\ \xi^2(\bar{x}, \bar{t}, \bar{u}) & = 2C\bar{t}, \\ \eta(\bar{x}, \bar{t}, \bar{u}) & = B - 2C\bar{u}. \end{cases} \tag{3.51}$$

Now, the first ordinary differential equation is,

$$3C\bar{x} = \frac{d\bar{x}}{d\epsilon_1}. \tag{3.52}$$

Solving eq. (3.52) we obtain,

$$\bar{x} = c_1 e^{3C\epsilon_1}, \tag{3.53}$$

where c_1 is the integration constant. Subjecting eq. (3.53) to Lie's condition, i.e., $\bar{x} = x|_{\epsilon_1=0}$, we get,

$$\bar{x} = xe^{3C\epsilon_1}. \tag{3.54}$$

The second equation is,

$$2C\bar{t} = \frac{d\bar{t}}{d\epsilon_1}, \tag{3.55}$$

solving eq. (3.55) and taking into consideration Lie's condition, we get

$$\bar{t} = te^{2C\epsilon_1}. \tag{3.56}$$

Similarly, solving the third ordinary differential equation subject to Lie's condition yields,

$$\bar{u} = ue^{-2C\epsilon_1}. \tag{3.57}$$

Therefore the Lie group[3] corresponding to the parameter ϵ_1 is,

$$T_{\epsilon_1}: \quad \bar{x} = xe^{3C\epsilon_1}, \quad \bar{t} = te^{2C\epsilon_1}, \quad \bar{u} = ue^{-2C\epsilon_1}. \tag{3.58}$$

Carrying out the same procedure for the remaining parameters with their corresponding Lie point symmetries result in the following additional one-parameter Lie groups

$$T_{\epsilon_2}: \quad \bar{x} = x + \epsilon_2, \quad \bar{t} = t, \quad \bar{u} = u, \tag{3.59}$$

$$T_{\epsilon_3}: \quad \bar{x} = x, \quad \bar{t} = t + \epsilon_3, \quad \bar{u} = u. \tag{3.60}$$

We will now proceed to demonstrate that our model in eq. (3.2) is invariant under the one-parameter Lie groups found above. As an example consider the dilation group in eq. (3.58),

$$T_{\epsilon_1}: \quad \bar{x} = xe^{3C\epsilon_1}, \quad \bar{t} = te^{2C\epsilon_1}. \quad \bar{u} = ue^{-2C\epsilon_1}.$$

Thus, the independent variables x, t and dependent variable u, transform as follows,

$$x = \bar{x}e^{-3C\epsilon_1}, \quad t = \bar{t}e^{-2C\epsilon_1}, \quad u = \bar{u}e^{2C\epsilon_1},$$

consider

---

[3] For a proof that the transformations found in this work, for an example, in eq. (3.58), indeed form a one-parameter group see the Appendix 5.

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial \bar{u} e^{2C\epsilon_1}}{\partial x} \\
&= e^{2C\epsilon_1} \frac{\partial \bar{u}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x} \\
&= e^{5C\epsilon_1} \frac{\partial \bar{u}}{\partial \bar{x}}.
\end{aligned}
$$

Using the adopted standard notation we have,

$$u_x = e^{5C\epsilon_1} \bar{u}_{\bar{x}},$$

following the same procedure, we find how other derivatives of the dependent variable u transform. Below, we provide a tabular representation of such transformations.

Original derivatives	Transformation	Transformed derivatives
u_x	$T_{\epsilon_1} : (x, t, u) \to (\bar{x}e^{-3C\epsilon_1}, \bar{t}e^{-2C\epsilon_1}, \bar{u}e^{2C\epsilon_1})$	$e^{5C\epsilon_1} \bar{u}_{\bar{x}}$
u_t		$e^{4C\epsilon_1} \bar{u}_{\bar{t}}$
u_{xx}		$e^{8C\epsilon_1} \bar{u}_{\bar{x}\bar{x}}$
u_{tt}		$e^{6C\epsilon_1} \bar{u}_{\bar{t}\bar{t}}$
u_{ttt}		$e^{8C\epsilon_1} \bar{u}_{\bar{t}\bar{t}\bar{t}}$

Table 3.3: Table of derivative transformations under T_{ϵ_1}

.

Thus, the transformed form of eq. (3.2) is

$$\bar{A}\bar{u}_{\bar{x}\bar{x}} - \bar{B}\bar{u}_{\bar{t}\bar{t}} + \bar{u}_{\bar{t}\bar{t}\bar{t}} + 2\bar{C}(\bar{u}_{\bar{t}})^2 + 2\bar{C}\bar{u}\bar{u}_{\bar{t}\bar{t}} = 0, \tag{3.61}$$

where $\bar{A} = A$, $\bar{B} = Be^{-2C\epsilon_1}$ and $\bar{C} = C$. Therefore our model is invariant under the Lie group of transformations in eq. (3.58).

3.5.2 Exact invariant solutions

For this part, we provide the exact solutions of the invariant sub-models constructed from the optimal system described in Section 3.3. It is worth noting that the invariant sub-model (3.40) is highly non-linear; therefore, it is not solved. We showcase the solutions categorized by different symmetries and their combinations. To fully understand the implications of the obtained invariant solutions, we represent them as 2D graphs and surface plots (3D graphs). The graphical simulations that follow represent the propagation of acoustic waves in mammalian blood tissue. The relationship between the physical acoustical parameters of blood tissue is vital for a number of reasons, including studying the dynamics of tissue propagation characteristics, creation and implementation of accurate simulations of acoustic propagation in human tissue, and effective design of quantitative ultrasonic imaging modalities and interpretation.

A tabular representation of the physical acoustical parameters of blood tissue which are utilized in the simulations that follow is presented below.

Acou. Par.	S.I. Unit	Value
Speed	$mm/\mu s$	1.584
Density	g/cm^3	1.060
Non-lin. par.		6.1
Atten. coef.	dB/cm	0.20

Table 3.4: Sound speed, density, attenuation, and non-linearity parameter values used for analysis. Source [58].

Case 1: X_2.

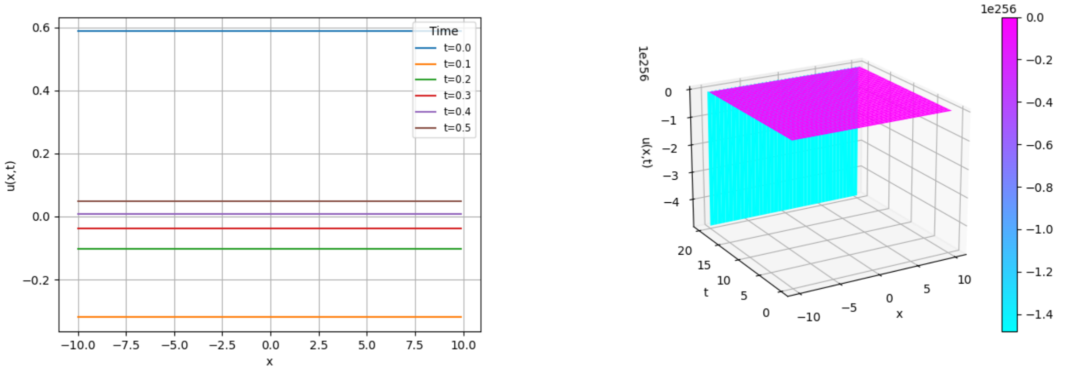
The associated invariant sub-model is given by eq. (3.42), the exact invariant solution is,

$$u(x,t) = \frac{c_1(2CBi'(\xi_0)c_1 + BBi(\xi_0)(Cc_1)^{\frac{2}{3}} + c_3(2CAi'(\xi_0)c_1 + BAi(\xi_0)(Cc_1)^{\frac{2}{3}}))}{2(Cc_1)^{\frac{5}{3}}(Bi(\xi_0) + Ai(\xi_0)c_3)}, \quad (3.62)$$

where $c_i, (i = 1, 2, 3)$ are constants of integration and $\xi_0 = \frac{B^2+4Ctc_1-4Cc_2}{4(Cc_1)^{\frac{2}{3}}}$.

From Figure 3.1, we observe a constant pressure profile for the majority of the simulated time and space followed by an abrupt decrease in acoustic pressure at the final time. However, if the pressure falls below the rest position and becomes negative, we classify this wave as a dark wave structure.

Figure 3.1: Pressure profile in 2D and 3D for eq. (3.62), assuming $c_i = 1$. This is for $-10 \leq x \leq 10$, and $0 \leq t \leq 20$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

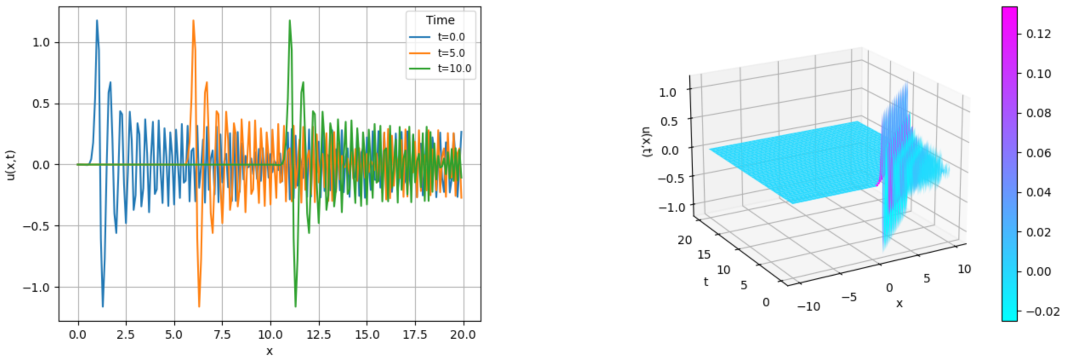
Case 2: $X_3 + \alpha X_2$.

We consider the invariant sub-model given in eq. (3.49), the associated invariant solution is,

$$u(x,t) = \frac{2^{\frac{2}{3}}\kappa Bi'(\xi_0)c_1 + \tau Bi(\xi_0)(-\kappa c_1)^{\frac{2}{3}} + (2^{\frac{2}{3}}\kappa Ai'(\xi_0)c_1 + \tau Ai(\xi_0)(-\kappa c_1)^{\frac{2}{3}})c_3}{\kappa(\kappa c_1)^{\frac{2}{3}}(Bi(\xi_0) + Ai(\xi_0)c_3)}, \quad (3.63)$$

where $\frac{2C}{\alpha} = \kappa$, $\frac{B}{\alpha} - \frac{A}{\alpha^3} = \tau$ and $c_i, (i = 1, 2, 3)$ are the constants of integration. Also, $\xi_0 = \frac{\tau^2 - 2(x-\alpha t)\kappa c_1 + 2\kappa c_2}{2^{\frac{4}{3}}(-\kappa c_1)^{\frac{2}{3}}}$.

Figure 3.2: Pressure profile in 2D and 3D for eq. (3.48), where $c_i = 1$. The simulation is for $-10 \leq x \leq 10$, $0 \leq t \leq 20$ and $\alpha = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

Figure 3.2 is a depiction of a periodic singular kink wave propagation with a shock in-front of the solitary wave. The travelling wave exhibits a repeating pattern of localised wave fronts.

Due to the wave having both negative and positive amplitude this wave qualifies as a bright-dark solitary wave. These types of wave structures have many applications in medical imaging techniques such as high intensity focused ultrasound [25,73].

We continue to seek more closed-form invariant solutions of eq. (3.49) by using the modified simple equation method in Section 2.2. Firstly, we rewrite our sub-model as

$$\mathcal{G}''' + \tau\mathcal{G}'' - \frac{\kappa}{2}(\mathcal{G}^2)'' = 0, \tag{3.64}$$

where $\frac{2C}{\alpha} = \kappa$, $\frac{B}{\alpha} - \frac{A}{\alpha^3} = \tau$. Integrating eq. (3.64) assuming the integration constants to be zero, we get

$$\mathcal{G}'' + \tau\mathcal{G}' - \kappa\mathcal{G}\mathcal{G}' = 0, \tag{3.65}$$

similarly, we integrate eq. (3.65) to get the desired form for this method, which is given by

$$\mathcal{G}' + \tau\mathcal{G} - \frac{\kappa}{2}\mathcal{G}^2 = 0. \tag{3.66}$$

Now assume eq. (3.66) admits the following solution

$$\mathcal{G} = \sum_{i=0}^{M} A_i \left(\frac{F'}{F}\right)^i, \tag{3.67}$$

by balancing $\mathcal{G}'$ and $\mathcal{G}^2$, we get that $M + 1 = 2M$, which gives $M = 1$, thus,

$$\mathcal{G} = A_0 + A_1 \frac{F'}{F,} \tag{3.68}$$

which implies,

$$\mathcal{G}' = A_1 \left(\frac{F''}{F} - \left(\frac{F'}{F}\right)^2\right). \tag{3.69}$$

Substituting eqs. (3.68) and (3.69) into eq. (3.66), we get

$$A_1\left(\frac{F''}{F}-\left(\frac{F'}{F}\right)^2\right)+\tau\left(A_0+A_1\frac{F'}{F}\right)-\frac{\kappa}{2}\left(A_0^2+A_1^2\left(\frac{F'}{F}\right)^2+2A_0A_1\frac{F'}{F}\right)=0. \quad (3.70)$$

Separating eq. (3.70) with respect to F^i, $(i=0,-1,-2)$, yields the following system of equations

$$F^0: \quad \tau A_0-\frac{\kappa}{2}A_0^2=0, \quad (3.71)$$

$$F^{-1}: \quad A_1F''+\tau A_1F'-\kappa A_0A_1F'=0, \quad (3.72)$$

$$F^{-2}: \quad -A_1(F')^2-\frac{\kappa}{2}A_1^2(F')^2=0. \quad (3.73)$$

Using eqs. (3.71) and (3.73) we obtain

$$A_0=0,\frac{2\tau}{\kappa}, \quad (3.74)$$

$$A_1=0,\frac{-2}{\kappa}. \quad (3.75)$$

According to Step. 2, in Section 2.2, the case $A_1=0$, is left out. We obtain the cases as follows:

Case 1: $A_0=0$ and $A_1=\frac{-2}{\kappa}$,

substituting the constants A_0 and A_1 into eq. (3.72), we get

$$F''+\tau F'=0, \quad (3.76)$$

which has the following solution,

$$F=-\frac{c_1}{\tau}e^{-J_2\tau}+c_2, \quad (3.77)$$

thus differentiating eq. (3.77), we obtain,

$$F'=c_1e^{-J_2\tau}. \quad (3.78)$$

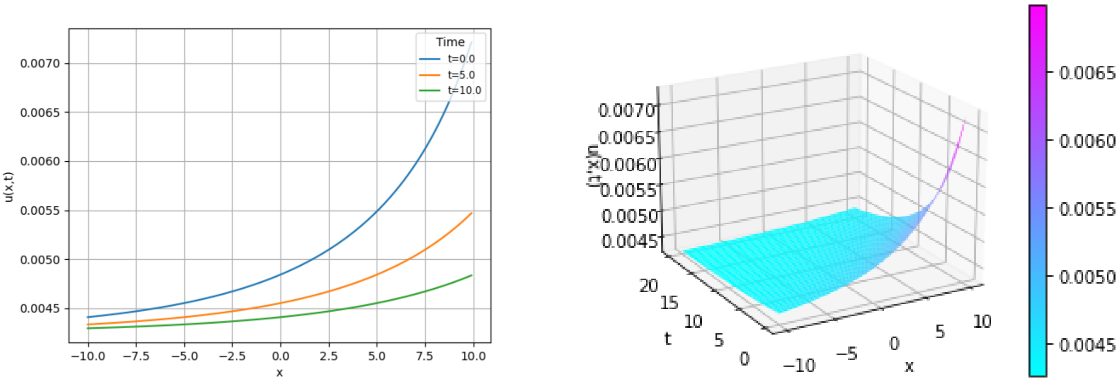
We substitute eqs. (3.77) and (3.78) into eq. (3.68) taking into account that $u(x,t)=\mathcal{G}(x-\alpha t)$

to get the solution

$$u(x,t) = \frac{-2}{\kappa}\left(\frac{c_1\tau}{\tau c_2 e^{\tau(x-\alpha t)} - c_1}\right), \tag{3.79}$$

where $c_i, (i = 1, 2)$ are constants of integration.

Figure 3.3: Pressure profile in 2D and 3D for eq. (3.79). This for $-10 \leq x \leq 10$, $0 \leq t \leq 20$ and $\alpha = 1$, such that $c_i = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

From Figure 3.3, at initial time, the acoustic pressure increases exponentially, but as time elapses, we observe a gradual decrease in the gradient of the exponential increase in acoustic pressure. This is considered as a kink waveform that has a feathery 3-D structure.

Case 2: $A_0 = \frac{2\tau}{\kappa}$ and $A_1 = \frac{-2}{\kappa}$.

Substituting the arbitrary constants A_0 and A_1 into eq. (3.72), we obtain the ordinary differential equation

$$F'' - \tau F' = 0, \tag{3.80}$$

whose solution is

$$F = \frac{c_1}{\tau} e^{J_2\tau} + c_2, \tag{3.81}$$

and derivative is,

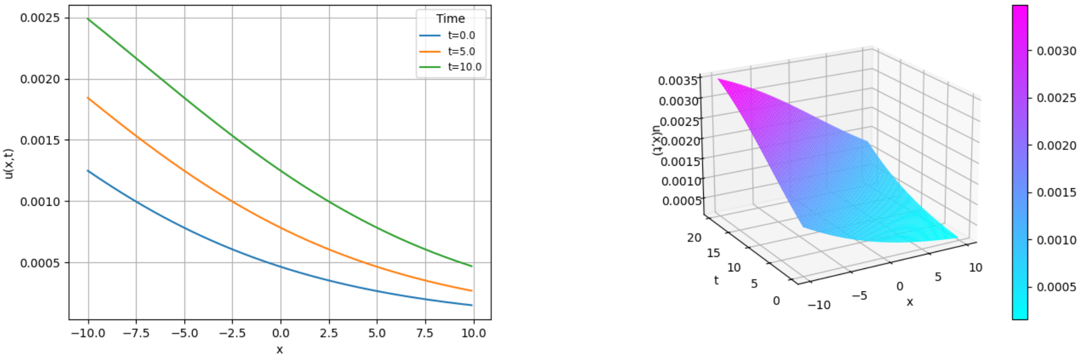
$$F' = c_1 e^{J_2\tau}. \tag{3.82}$$

Substitute eqs. (3.81) and (3.82) into eq. (3.68) to get the exact solution as

$$u(x,t) = \frac{2\tau}{\kappa} - \frac{2}{\kappa}\left(\frac{c_1 \tau e^{\tau(x-\alpha t)}}{c_1 e^{\tau(x-\alpha t)} + \tau c_2}\right), \tag{3.83}$$

where $c_i (i = 1, 2)$ are arbitrary constants of integration.

Figure 3.4: Pressure profile in 2D and 3D for eq. (3.83) where $c_i = 1$. This is for $-10 \leq x \leq 10$, $0 \leq t \leq 20$ and $\alpha = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

From Figure 3.4, we observe the opposite waveform Figure 3.3. The gradient of the exponential decay in acoustic pressure is increasing as time elapses. This could signify an absorption of acoustical energy by the medium.

3.6 Conclusion

In this chapter, we performed Lie symmetry analysis for the Westervelt equation, which is represented by eq. (3.1). We found the Lie point symmetries which were utilised to generate the one parameter Lie groups by solving Lie's equations. Moreover, we constructed a family of exact closed form invariant solutions from the reduced invariant sub-models. Lastly, we represented the graphical solutions which are comprised of a number of wave structures ranging from kinks, feathery, bright, dark and periodic. In Figure 3.2, we observe an amalgamation of numerous waveforms in a single structure, i.e., a periodic kink wave that exhibits both dark and bright forms.

Chapter 4

Lie symmetry analysis of the Westervelt equation: Model II

In Chapter 3, we explored Model I (3.1), which has the third-order temporal derivative to account for the acoustic energy loss. We consider a variation to Model I where the linear wave relation is incorporated into the dissipative or loss term to account for the wave's dispersion and energy loss due to damping in a Newtonian fluid[1]. This is encapsulated by the partial differential equation below,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 u}{\partial t^2} + \frac{\delta}{c_0^2}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial t}\right) + \frac{\beta}{\rho c_0^4}\frac{\partial^2 u^2}{\partial t^2} = 0. \tag{4.1}$$

The resulting Westervelt model in eq. (4.1) shall be referred to as Model II. The Lie point symmetries of eq. (4.1) are found and utilized to find an optimal system, then, similarity reductions are performed to obtain invariant sub-models. Subsequently, the one-parameter Lie groups are generated, followed by the presentation of the exact solutions to the invariant sub-models together with their graphs.

---

[1] Since the viscosity of a Newtonian fluid depends on time and space, this ensures that the dissipation term has both space and time dependency.

4.1 Determining equations and infinitesimal generator

Similarly as we did for Model I, we rewrite the model in eq. (4.1) as,

$$u_{txx} = Bu_{tt} - Au_{xx} - 2Cu_t^2 - 2Cuu_{tt} \tag{4.2}$$

where $A = \frac{c_o^2}{\delta}$, $B = \frac{1}{\delta}$ and $C = \frac{\beta}{\rho\delta c_0^2}$, such that $\delta \neq 0$, $c_0 \neq 0$, $\beta \neq 0$, and $\rho \neq 0$. The vector field,

$$X = \xi^1(x,t,u)\frac{\partial}{\partial x} + \xi^2(x,t,u)\frac{\partial}{\partial t} + \eta(x,t,u)\frac{\partial}{\partial u}, \tag{4.3}$$

is an infinitesimal generator of eq. (4.1), provided,

$$\mathbf{Pr}^{[3]}X(Au_{xx} - Bu_{tt} + u_{txx} + 2C(u_t)^2 + 2Cuu_{tt})|_{(4.2)} = 0,$$

where the third prolongation of X is given as,

$$\begin{aligned}\mathbf{Pr}^{[3]}X = {} & \xi^1\partial_x + \xi^2\partial_t + \eta\partial_u + \zeta_t\partial_{u_t} + \zeta_x\partial_{u_x} + \zeta_{tt}\partial_{u_{tt}} + \zeta_{tx}\partial_{u_{tx}} + \zeta_{xx}\partial_{u_{xx}} + \zeta_{ttt}\partial_{u_{ttt}} \\ & + \zeta_{ttx}\partial_{u_{ttx}} + \zeta_{txx}\partial_{u_{txx}} + \zeta_{xxx}\partial_{u_{xxx}},\end{aligned}$$

such that the coefficients ζ_{ijk} are given by eqs. (3.4a) to (3.4e) and (3.4g) to (3.4i).

The third prolongation of X acting on eq. (1.10) subject to the condition in eq. (4.2), yields

$$2C\eta u_{tt} + 4C\zeta_t u_t - B\zeta_{tt} + 2C\zeta_{tt}u + A\zeta_{xx} + \zeta_{txx} = 0, \tag{4.4}$$

which upon expansion of the ζ's becomes

$$
\begin{aligned}
&2C\eta u_{tt} - 2Bu_{tt}u_x\xi^1_u + 4Cuu_{tt}u_x\xi^1_u - 3u_{xt}u_{xx}\xi^1_u - 2Au_xu_{xt}\xi^2_u - 2u^2_{xt}\xi^2_u - 2u_xu_{xtt}\xi^2_u \\
&- u_{tt}u_{xx}\xi^2_u + Au^2_x\eta_{uu} + 2u_xu_{xt}\eta_{uu} - Au^3_x\xi^1_{uu} - 3u^2_xu_{xt}\xi^1_{uu} + (B-2Cu)u^3_t\xi^2_{uu} \\
&- u_{tt}u^2_x\xi^2_{uu} + 2Bu_{xt}\xi^1_t - 4Cuu_{xt}\xi^1_t - u_{xxx}\xi^1_t + Bu_{tt}\xi^2_t - 2Cuu_{tt}\xi^2_t + Au_{xx}\xi^2_t \\
&+ u_{xx}\eta_{tu} - 3u_xu_{xx}\xi^1_{tu} - 2u_xu_{xt}\xi^2_{tu} + u^2_x\eta_{tuu} - u^3_x\xi^1_{tuu} - B\eta_{tt}2Cu\eta_{tt} + Bu_x\xi^1_{tt} \\
&- 2Cuu_x\xi^1_{tt} - 2Bu_{tt}\xi^1_x + 4Cuu_{tt}\xi^1_x - 2Au_{xt}\xi^2_x - 2u_{xtt}\xi^2_x - 2Au_x\eta_{xu} + 2u_{xt}\eta_{xu} \\
&- 2Au^2_x\xi^1_{xu} - 4u_xu_{xt}\xi^1_{xu} - 2u_{tt}u_x\xi^2_{xu} - 2u_{xx}\xi^1_{xt} - 2u_{xt}\xi^2_{xt} + 2u_x\eta_{xtu} - 2u^2_x\xi^1_{xtu} \\
&+ A\eta_{xx} - Au_x\xi^1_{xx} - u_{xt}\xi^1_{xx} - u_{tt}\xi^2_{xx} + u^2_t(2C\eta_u - B\eta_{uu} + 2Cu\eta_{uu} - u_{xx}\xi^2_{uu} \\
&- u^2_x\xi^2_{uuu} - 2C\xi^2_t + B\xi^2_{tu} - 4Cu\xi^2_{tu} + 4C\xi^1_x + u_x(2C\xi^1_u + (B-2Cu)\xi^1_{uu} - 2\xi^2_{xuu}) \\
&- \xi^2_{xxu}) + \eta_{xxt} - u_x\xi^1_{xxt} + u_t(-u_{xxx}\xi^1_u + Bu_{tt}\xi^2_u - 2Cuu_{tt}\xi^2_u + Au_{xx}\xi^2_u + u_{xx}\eta_{uu} \\
&- 3u_xu_{xx}\xi^1_{uu} - Au^2_x\xi^2_{uu} + u^2_x\eta_{uuu} - u^3_x\xi^1_{uuu} + 4C\eta_t - 4Cu_x\xi^1_t - 2B\eta_{tu} + 4Cu\eta_{tu} \\
&+ 2Bu_x\xi^1_{tu} - 4Cuu_x\xi^1_{tu} - u_{xx}\xi^2_{tu} - u^2_x\xi^2_{tuu} + B\xi^2_{tt} - 2Cu\xi^2_{tt} - 2u_{xx}\xi^1_{xu} - 2Au_x\xi^2_{xu} \\
&+ 2u_{xt}((B-2Cu)\xi^1_u - 2(u_x\xi^2_{uu} + \xi^2_{xu})) + 2u_x\xi^1_{xuu} - 2u^2_x\xi^1_{xuu} - 2u_x\xi^2_{xtu} - A\xi^2_{xx} \\
&+ \eta_{xxu} - u_x\xi^1_{xxu} - \xi^2_{xxt}) = 0.
\end{aligned}
\tag{4.5}
$$

Separating eq. (4.5) with respect to powers and products of the derivatives of $u(x,t)$, we get the following overdetermined system of linear homogeneous PDEs

$$\xi^1_u = 0, \tag{4.6a}$$

$$\xi^2_u = 0, \tag{4.6b}$$

$$\eta_{uu} = 0, \tag{4.6c}$$

$$\xi^1_t = 0, \tag{4.6d}$$

$$\xi^2_x = 0, \tag{4.6e}$$

$$A\xi^2_t + \eta_{tu} = 0, \tag{4.6f}$$

$$2\eta_{xu} - \xi^1_{xx} = 0, \tag{4.6g}$$

$$2C\eta + (-B + 2Cu - 2(-B + 2Cu))\xi^2_t + (-2B + 4Cu)\xi^1_x = 0, \tag{4.6h}$$

$$2C\eta - 2C\xi^2_t + 4C\xi^1_x = 0, \tag{4.6i}$$

$$(-B + 2Cu)\xi^1_{tt} + A\xi^1_{xx} + \xi^1_{xxt} = 0, \tag{4.6j}$$

$$2A\xi^1_{xu} + 2\xi^1_{xtu} - A\xi^1_{xx} = 0, \tag{4.6k}$$

$$4C\xi^1_t + 2(-B + 2Cu)\xi^1_{tu} + (B - 2Cu)\xi^2_{tt} + \xi^1_{xxu} = 0. \tag{4.6l}$$

From eqs. (4.6a) to (4.6e), we get

$$\xi^1 = \xi^1(x), \tag{4.7}$$

$$\xi^2 = \xi^2(t), \tag{4.8}$$

$$\eta = uf(x,t) + g(x,t). \tag{4.9}$$

We update our system by taking into consideration eqs. (4.7) to (4.9), which yield the determining equations,

$$A\xi^2_t + f_t = 0, \tag{4.10a}$$

$$-\xi^1_{xx} + 2f_x = 0, \tag{4.10b}$$

$$2Cuf + 2Cg - 2B\xi^1_x + 4Cu\xi^1_x + B\xi^2_t - 2Cu\xi^2_t = 0, \tag{4.10c}$$

$$2Cf + 4C\xi^1_x - 2C\xi^2_t = 0, \tag{4.10d}$$

$$-Buf_{tt} + 2Cu^2 f_{tt} - Bg_{tt} + 2Cug_{tt} + Auf_{xx} + Ag_{xx} + uf_{xxt} + g_{xxt} = 0, \tag{4.10e}$$

$$A\xi^1_{xx} + 2Af_x + 2f_{xt} = 0, \tag{4.10f}$$

$$B\xi^2_{tt} - 2Cu\xi^2_{tt} - 2Bf_t + 8Cuf_t + 4Cg_t + f_{xx} = 0. \tag{4.10g}$$

From eq. (4.10d), we get

$$f(x,t) = \xi^2_t - 2\xi^1_x. \tag{4.11}$$

Substituting eq. (4.11) into eq. (4.9) results in the following updated system of determining equations

$$A\xi^2_t + \xi^2_{tt} = 0, \tag{4.12a}$$

$$-5\xi^1_{xx} = 0, \tag{4.12b}$$

$$2Cg - 2B\xi^1_x + B\xi^2_t = 0, \tag{4.12c}$$

$$-2Au\xi^1_{xxx} - Bu\xi^2_{ttt} + 2Cu^2\xi^2_{ttt} - Bg_{tt} + 2Cug_{tt} + Ag_{xx} + g_{xxt} = 0, \tag{4.12d}$$

$$-5A\xi^1_{xx} = 0, \tag{4.12e}$$

$$-B\xi^2_{tt} + 6Cu\xi^2_{tt} - 2\xi^1_{xxx} + 4Cg_t = 0. \tag{4.12f}$$

Considering eq. (4.12c), we get

$$g(x,t) = \frac{B}{C}\xi^1_x - \frac{B}{2C}\xi^2_t. \tag{4.13}$$

Substituting eq. (4.13) into the updated eq. (4.9), we obtain the system

$$A\xi^2_t + \xi^2_{tt} = 0, \tag{4.14a}$$

$$-5\xi^1_{xx} = 0, \tag{4.14b}$$

$$\frac{AB}{C}\xi^1_{xxx} - 2Au\xi^1_{xxx} + \frac{B^2}{2C}\xi^2_{ttt} - 2Bu\xi^2_{ttt} + 2Cu^2\xi^2_{ttt} = 0, \tag{4.14c}$$

$$-5A\xi^1_{xx} = 0, \tag{4.14d}$$

$$-3B\xi^2_{tt} + 6Cu\xi^2_{tt} - 2\xi^1_{xxx} = 0. \tag{4.14e}$$

From eqs. (4.14b) and (4.14d), we have

$$\xi^1(x) = xk_1 + k_2, \tag{4.15}$$

differentiating eq. (4.15) thrice with respect to x, we get

$$\xi^1_{xxx} = 0. \tag{4.16}$$

From eq. (4.14a), we observe the relation

$$\xi^2_{tt} = -A\xi^2_t. \tag{4.17}$$

Substituting eqs. (4.16) and (4.17) into eq. (4.14e), we obtain

$$3AB\xi^2_t - 6ACu\xi^2_t = 0. \tag{4.18}$$

Separating eq. (4.18) with respect powers of $u(x,t)$,

$$u^0 : \xi^2_t = 0, \tag{4.19a}$$

$$u^1 : \xi^2_t = 0. \tag{4.19b}$$

From eqs. (4.11), (4.15), (4.19a) and (4.19b), we get the infinitesimals as

$$\xi^1(x) = xk_1 + k_2, \tag{4.20a}$$

$$\xi^2(t) = k_3, \tag{4.20b}$$

$$\eta(u) = \left(\frac{B}{C} - 2u\right) k_1. \tag{4.20c}$$

The symmetry generator (4.3), is given by

$$X = (xk_1 + k_2)\partial_x + k_3\partial_t + \left(\frac{B}{C} - 2u\right) k_1\partial_u. \tag{4.21}$$

4.2 Lie point symmetries

Model II admits a three dimensional Lie algebra, i.e, $\mathbb{L}_3$, with the following basis,

$$X_1 = x\partial_x + \left(\frac{B}{C} - 2u\right)\partial_u \approx Cx\partial_x + (B - 2Cu)\partial_u, \tag{4.22a}$$

$$X_2 = \partial_x, \tag{4.22b}$$

$$X_3 = \partial_t. \tag{4.22c}$$

4.3 Optimal system of sub-algebras

In this section, we construct one-dimensional optimal system for the Lie algebra with basis as eqs. (4.22b) to (4.22c).

A summarized presentation of the commutations and adjoint representations of the Lie point symmetries in eqs. (4.22a) to (4.22c), is provided in Tables 4.1 and 4.2.

Likewise, for the construction of the one-dimensional optimal sub-algebra for the symmetry group $\langle X_1, X_2, X_3\rangle$, we consider the general symmetry operator

$$X = a_1X_1 + a_2X_2 + a_3X_3, \tag{4.23}$$

$\nearrow [X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-CX_2$	0
X_2	CX_2	0	0
X_3	0	0	0

Table 4.1: Lie Algebras

$Ad(e^{\epsilon X_i})X_j$	X_1	X_2	X_3
X_1	X_1	$e^{\epsilon C}X_2$	X_3
X_2	$X_1 - \epsilon CX_2$	X_2	X_3
X_3	X_1	X_2	X_3

Table 4.2: Adjoint representation

where a_i, $(i = 1, 2, 3)$ are arbitrary constants. We begin by assuming $a_1 \neq 0$, now, re-scaling a_1 such that $a_1 = 1$ to obtain,

$$X = a_2X_2 + a_3X_3 + X_1. \tag{4.24}$$

Acting on eq. (4.24) by $Ad(e^{\frac{a_2}{C}X_2})$ to eliminate a_2X_2, we get,

$$\begin{aligned} X' &= Ad(e^{\frac{a_2}{C}X_2})X \\ &= Ad(e^{\frac{a_2}{C}X_2})(a_2X_2 + a_3X_3 + X_1) \\ &= a_2Ad(e^{\frac{a_2}{C}X_2})X_2 + a_3Ad(e^{\frac{a_2}{C}X_2})X_3 + Ad(e^{\frac{a_2}{C}X_2})X_1 \\ &= a_2X_2 + a_3X_3 + X_1 - C(\frac{a_2}{C})X_2 \\ &= a_3X_3 + X_1. \end{aligned} \tag{4.25}$$

From Table 4.2, we see that eq. (4.25) cannot be reduced to a simpler form, thus depending on the sign of a_3 we can make the coefficients of X_3 to be $+1, -1$. Thus, the one-dimensional sub-algebra generated by X, such that $a_1 \neq 0$, is equivalent to a sub-algebra spanned by $X_1 + \gamma X_3;\ \gamma \pm 1$. For $a_3 > 0 : X = X_1 + X_3, a_3 < 0 : X = X_1 - X_3$. Also, for $a_3 = 0 : X = X_1$. Therefore, any one-dimensional sub-algebra generated by X, such that $a_1 \neq 0$ and $a_3 = 0$, is equivalent to a sub-algebra spanned by X_1

Secondly, taking $a_1 = 0$ and $a_2 \neq 0$ such that $a_2 = 1$, we get

$$X = X_2 + a_3 X_3. \tag{4.26}$$

As per Table 4.2, we observe that eq. (4.26) cannot be reduced any further, therefore varying a_3 we can make the coefficients of X_3 to be $+1, -1$. Hence, any one-dimensional sub-algebra generated by X, where $a_2 \neq 0$ and $a_1 = 0$, is equivalent to a sub-algebra spanned by $X_2 + \Lambda X_3;\ \ \Lambda = \pm 1$. Thus, $a_3 > 0 : X = X_2 + X_3, a_3 < 0 : X = X_2 - X_3$. For $a_3 = 0 : X = X_2$, therefore, any one-dimensional sub-algebra generated by X, such that $a_2 \neq 0$ and $a_3 = 0$, is equivalent to a sub-algebra spanned by X_2.

Lastly, taking $a_1 = a_2 = 0$ and $a_3 \neq 0$ such that $a_3 = 1$, we get, $X = X_3$. Implying, any one-dimensional sub-algebra generated by X, where $a_3 \neq 0$ and $a_2 = a_1 = 0$, is the same as a sub-algebra generated by X_3. Therefore the one-dimensional optimal system of sub-algebras is spanned by the set

$$\{X_1, X_2, X_3, X_1 - X_3, X_1 + X_3, X_2 - X_3, X_2 + X_3\}. \tag{4.27}$$

4.4 Symmetry reductions and invariant sub-models

In this section, we utilize the one-dimensional optimal system of sub-algebras found in Section 4.3 to perform similarity reductions, which yield invariant sub-models. This means we employ the symmetries together with their respective combinations to transform our governing model from a partial differential equation into ordinary differential equations.

4.4.1 Invariance under X_1.

The corresponding characteristic equation is given by,

$$\frac{dx}{Cx} = \frac{dt}{0} = \frac{du}{B - 2Cu}. \tag{4.28}$$

Considering the second ratio of eq. (4.28) we get $t = a_1$, as the first invariant. From the first and last ratios of eq. (4.28), we obtain $x^2\left(u-\frac{B}{2C}\right)=a_2$, as the second invariant. Taking into account the condition that $f(a_1)=a_2$, we get that

$$u(x,t)=\frac{B}{2C}+\frac{f}{x^2}. \tag{4.29}$$

Upon substituting eq. (4.29) into eq. (4.2) results in the following ordinary differential equation

$$6Af+6f'+2C(f')^2+2Cff''=0, \tag{4.30}$$

where $f'=\frac{df}{dt}$ and $f''=\frac{d^2f}{dt^2}$.

4.4.2 Invariance under X_2.

The associated characteristic equation is,

$$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0}. \tag{4.31}$$

From eq. (4.31) we get $t=p_1$ as the first invariant and $u=p_2$ as the second invariant. Now taking into consideration the invariant solution $h(p_1)=p_2$, i.e., $h(t)=u(x,t)$ and substituting it into eq. (4.2) yields,

$$-Bh''+C(2(h')^2+2hh'')=0, \tag{4.32}$$

where $h'=\frac{dh}{dt}$ and $h''=\frac{d^2h}{dt^2}$.

4.4.3 Invariance under X_3.

The characteristic equation associated with X_3 is

$$\frac{dx}{0}=\frac{dt}{1}=\frac{du}{0}. \tag{4.33}$$

From eq. (4.33) we get $x=n_1$, $u=n_2$ as the invariants, as such, the invariant solution is $g(n_1)=n_2$, i.e, $g(x)=u(x,t)$, upon substituting into eq. (4.2) yields the ordinary differential equation below

$$Ag'' = 0. \tag{4.34}$$

From eq. (4.34), we get

$$g'' = 0, \tag{4.35}$$

where $g'' = \frac{d^2g}{dx^2}$. The associated invariant solution obtained by solving eq. (4.35) is

$$u(x,t) = c_1x + c_2, \tag{4.36}$$

where, c_i, $(i = 1, 2)$, are the integration constants.

4.4.4 Invariance under $X = X_1 + \gamma X_3$.

The parameter $\gamma = \pm 1$ and the associated characteristic equation is

$$\frac{dx}{Cx} = \frac{dt}{\gamma} = \frac{du}{B - 2Cu}. \tag{4.37}$$

Considering the first and second ratios of eq. (4.37), we have $k_1 = \frac{x}{e^{\frac{C}{\gamma}t}}$ as the first invariant. Also, from the second and last ratios of eq. (4.37) we get the following first-order linear differential equation

$$\frac{du}{dt} + \frac{2C}{\gamma}u - \frac{B}{\gamma} = 0. \tag{4.38}$$

Solving eq. (4.38), we get, $k_2 = e^{\frac{2Ct}{\gamma}}\left(u - \frac{B}{2C}\right)$, as the second invariant. Thus, the invariant solution is $v(k_1) = k_2$, which leads to the general solution,

$$u(x,t) = \frac{B}{2C} + \frac{v}{e^{\frac{2C}{\gamma}t}}. \tag{4.39}$$

Upon substituting eq. (4.39) into eq. (4.2), we get the following ordinary differential equation

$$\frac{16C^3}{\gamma^2}v^2 + \frac{18C^3k_1}{\gamma^2}vv' + \frac{2C^3k_1^2}{\gamma^2}(v')^2 + Av'' - \frac{4C^3}{\gamma}v'' + \frac{2C^3k_1^2}{\gamma^2}vv'' - \frac{Ck_1}{\gamma}v''' = 0, \quad (4.40)$$

where $v' = \frac{dv}{dk_1}$, $v'' = \frac{d^2v}{dk_1^2}$ and $v''' = \frac{d^3v}{dk_1^3}$.

4.4.5 Invariance under $X = X_3 + \alpha X_2$.

From the described symmetry, the characteristic equation is given by

$$\frac{dx}{\alpha} = \frac{dt}{1} = \frac{du}{0}. \quad (4.41)$$

From eq. (4.41), we have $u = s_1$ as the first invariant and $x - \alpha t = s_2$ as the second invariant. The invariant solution is given by $g(s_2) = s_1$, i.e., $u(x,t) = g(x - \alpha t)$. Upon substitution into eq. (1.10), we get the ordinary differential equation

$$g''' - \left(\frac{A}{\alpha} - B\alpha\right) g'' - 2C\alpha(g')^2 - 2C\alpha g g'' = 0, \quad (4.42)$$

where $g' = \frac{dg}{ds_2}$, $g'' = \frac{d^2g}{ds_2^2}$ and $g''' = \frac{d^3g}{ds_2^3}$.

4.5 Results and Discussions

In this section, we present results and discussions for Model II. We begin by the construction of one-parameter Lie groups. We demonstrate that the model is invariant under the associated one-parameter Lie groups. Furthermore, we present the analytical solutions of the invariant submodels in Section 4.4. The graphical solutions are presented followed by a brief discussion. For the simulations in this section, the physical acoustical parameters are found in Table 3.4.

4.5.1 One-parameter Lie groups

The associated one-parameter Lie groups for some vector parameter $< \delta_1, \delta_2, \delta_3 >$, are

$$T_{\delta_1} : \bar{x} = xe^{C\delta_1}, \quad \bar{t} = t, \quad \bar{u} = ue^{-2C\delta_1}. \tag{4.43}$$

$$T_{\delta_2} : \bar{x} = x + \delta_2, \quad \bar{t} = t, \quad \bar{u} = u. \tag{4.44}$$

$$T_{\delta_3} : \bar{x} = x, \quad \bar{t} = t + \delta_3, \quad \bar{u} = u. \tag{4.45}$$

Below we provide a table that illustrates how the derivatives of $u(x,t)$ transform into $\bar{u}(\bar{x},\bar{t})$ under the one-parameter Lie group in eq. (4.43).

Original derivatives	Transformation	Transformed derivatives
u_x	$T_{\delta_1} : (x,t,u) \to (\bar{x}e^{-C\delta_1}, \bar{t}, \bar{u}e^{2C\delta_1})$	$e^{3C\delta_1}\bar{u}_{\bar{x}}$
u_t		$e^{2C\delta_1}\bar{u}_{\bar{t}}$
u_{xx}		$e^{4C\delta_1}\bar{u}_{\bar{x}\bar{x}}$
u_{tt}		$e^{2C\delta_1}\bar{u}_{\bar{t}\bar{t}}$
u_{txx}		$e^{4C\delta_1}\bar{u}_{\bar{t}\bar{x}\bar{x}}$

Table 4.3: Table of derivative transformations under T_{δ_1}

.

Thus, the transformed form of eq. (4.2) is

$$\bar{A}\bar{u}_{\bar{x}\bar{x}} - \bar{B}\bar{u}_{\bar{t}\bar{t}} + \bar{u}_{\bar{x}\bar{x}\bar{t}} + 2\bar{C}(\bar{u}_{\bar{t}})^2 + 2\bar{C}\bar{u}\bar{u}_{\bar{t}\bar{t}} = 0, \tag{4.46}$$

where $\bar{A} = A$, $\bar{B} = Be^{-2C\delta_1}$ and $\bar{C} = C$. Therefore Model II is invariant under the Lie group of transformations in eq. (4.43). Carrying out the same procedure with Lie group of transformations in eqs. (4.44) and (4.45), we find that the invariance criterion is satisfied.

4.5.2 Exact invariant solutions

For this part we provide exact solutions of the invariant sub-models constructed from the optimal system of sub-algebras found in Section 4.3. We solve the invariant sub-models to obtain exact solutions of the aforementioned model. We provide graphs corresponding to some analytical solutions followed by discussions. The invariant solutions of eqs. (4.30) and (4.40), could not be found.

Case 1: X_2.

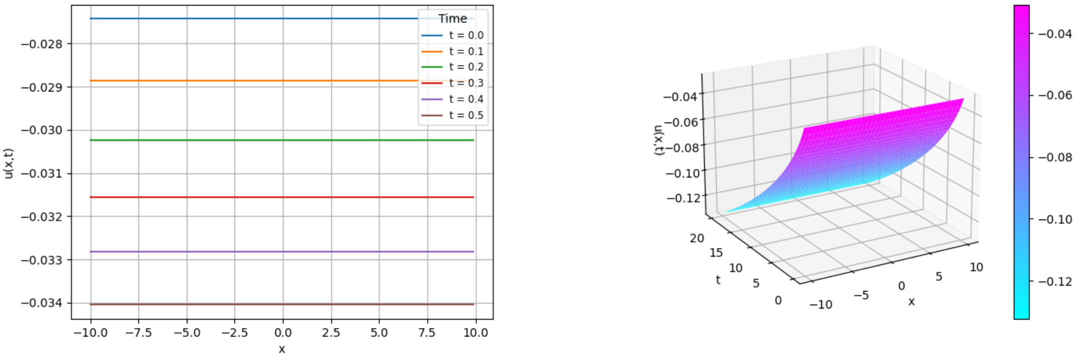
The associated invariant sub-model given by eq. (4.32), upon solving, we get two exact solutions given below,

$$u_1(x,t) = \frac{B - \sqrt{B^2 - 4Ctc_1 - 4Cc_1c_2}}{2C}, \tag{4.47}$$

$$u_2(x,t) = \frac{B + \sqrt{B^2 - 4Ctc_1 - 4Cc_1c_2}}{2C}, \tag{4.48}$$

where the constants of integration are given by c_i, $(i = 1, 2)$.

Figure 4.1: Pressure profile in 2D and 3D for eq. (4.47) where $-10 \leq x \leq 10$ and $0 \leq t \leq 20$.

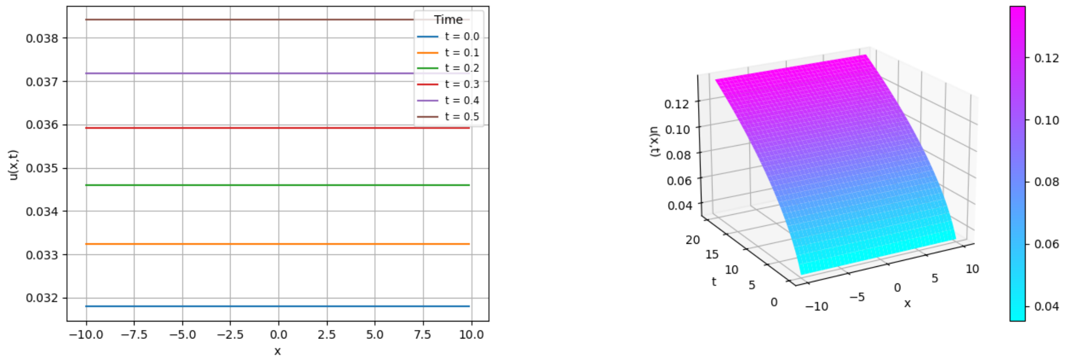


(a) 2D-Pressure graph

(b) 3D-Pressure graph

From Figure 4.1, we observe that for every time increment the acoustic pressure monotonically decreases while keeping a constant levels in space.

Figure 4.2: Pressure profile in 2D and 3D for eq. (4.48). This is for $-10 \leq x \leq 10$ and $0 \leq t \leq 20$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

We observe that acoustic pressure increases monotonically for the duration of the simulation. In particular, the constant pressure levels in space steadily increase in height as time evolves. Considering Figures 4.1b and 4.2b, we observe a rotational symmetry about the spatial axis between the two solutions.

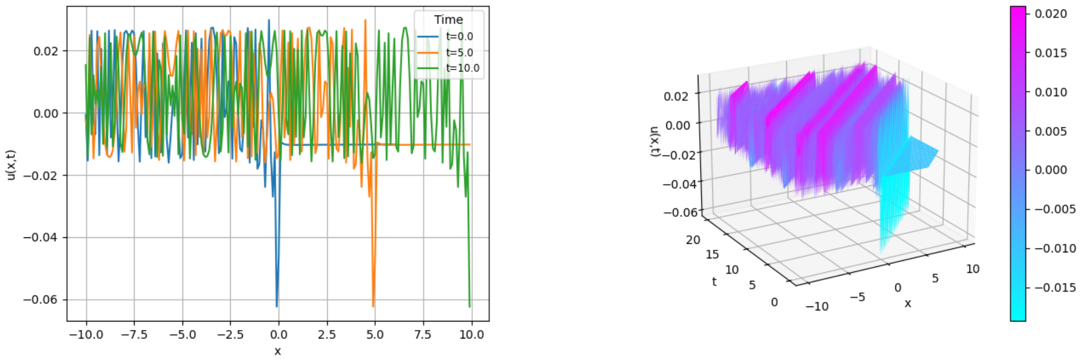
Case 2: $X_2 + \alpha X_3$.

Similarly, this linear combination produces a sub-model that admits the travelling wave solution, provided in eq. (4.42). The exact invariant solution is,

$$u(x,t) = \frac{-c_1(2\kappa Bi'(\xi_0)c_1 - \tau Bi'(\xi_0)(-\kappa c_1)^{\frac{2}{3}} + (2\kappa Ai'(\xi_0)c_1 - \tau Ai(\xi_0)(-\kappa c_1)^{\frac{2}{3}})c_3)}{2(-\kappa c_1)^{\frac{5}{3}}(Bi(\xi_0) + Ai(\xi_0)c_3)}, \tag{4.49}$$

where $\xi_0 = \frac{\tau^2 + 4\kappa(-(x-\alpha t)+c_2)}{4(-\kappa c_1)^{\frac{2}{3}}}$, $\tau = \frac{A}{\alpha} - B\alpha$ and $\kappa = \frac{C}{\alpha}$. Also, the constants of integration are represented by c_i, $(i = 1, 2, 3)$.

Figure 4.3: Pressure profile in 2D and 3D for eq. (4.49) where $-10 \leq x \leq 10$, $0 \leq t \leq 20$, $c_i, (i = 1, 2, 3) = 1$ and $\alpha = 1$. Also, $c_i = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

From Figure 4.3, we have a periodic kink waveform with a shock behind the solitary wave. We also note that the lengthiest wave humps have negative amplitude, thus this depicts a dark solitary waveform. These have applications in medical imaging [51], acoustic cavitation [55], e.t.c.

Also, we use the modified simple equation method to construct additional exact solutions for eq. (4.42), we rewrite our sub-model as

$$\mathcal{G}''' - \tau\mathcal{G}'' - \kappa(\mathcal{G}^2)'' = 0, \tag{4.50}$$

where $\tau = \frac{A}{\alpha} - B\alpha$ and $\kappa = C\alpha$. Integrating eq. (4.50) (while keeping the integration constants zero), we get

$$\mathcal{G}'' - \tau\mathcal{G}' - 2\kappa\mathcal{G}\mathcal{G}' = 0, \tag{4.51}$$

then, we integrate eq. (4.51) to get the desired form for this method, which is given by

$$\mathcal{G}' - \tau\mathcal{G} - \kappa\mathcal{G}^2 = 0. \tag{4.52}$$

Now, assume eq. (4.52) admits the following solution

$$\mathcal{G} = \sum_{i=0}^{M} A_i \left(\frac{F'}{F}\right)^i, \tag{4.53}$$

we find the value of M is 1, hence

$$\mathcal{G} = A_0 + A_1 \frac{F'}{F}, \tag{4.54}$$

which implies that

$$\mathcal{G}' = A_1 \left(\frac{F''}{F} - \left(\frac{F'}{F}\right)^2\right), \tag{4.55}$$

substituting eqs. (4.54) and (4.55) into eq. (4.52), we get

$$A_1 \left(\frac{F''}{F} - \left(\frac{F'}{F}\right)^2\right) - \tau \left(A_0 + A_1 \frac{F'}{F}\right) - \kappa \left(A_0^2 + A_1^2 \left(\frac{F'}{F}\right)^2 + 2A_0 A_1 \frac{F'}{F}\right) = 0. \tag{4.56}$$

Separating eq. (4.56) with respect to $F^i, (i = 0, -1, -2)$, yields the following system of equations

$$\begin{aligned} F^0 : &\quad -\tau A_0 - \kappa A_0^2 = 0, & (4.57)\\ F^{-1} : &\quad A_1 F'' - \tau A_1 F' - 2\kappa A_0 A_1 F' = 0, & (4.58)\\ F^{-2} : &\quad -A_1 (F')^2 - \kappa A_1^2 (F')^2 = 0. & (4.59) \end{aligned}$$

From eqs. (4.57) and (4.59), we get that

$$\begin{aligned} A_0 &= 0, \frac{-\tau}{\kappa}, & (4.60)\\ A_1 &= 0, \frac{-1}{\kappa}. & (4.61) \end{aligned}$$

Two cases arise from eqs. (4.60) and (4.61), namely $A_0 = 0$, $A_1 = \frac{-1}{\kappa}$ and $A_0 = \frac{-\tau}{\kappa}$, $A_1 = \frac{-1}{\kappa}$.

Case 1:$A_0 = 0$, $A_1 = \frac{-1}{\kappa}$

Taking into consideration the given constants A_0 and A_1, eq. (4.58) becomes,

$$F'' - \tau F' = 0, \tag{4.62}$$

upon solving eq. (4.62) we obtain,

$$F = c_1 \frac{e^{s_2 \tau}}{\tau} + c_2. \tag{4.63}$$

Differentiating eq. (4.63) with respect to s_2 we get,

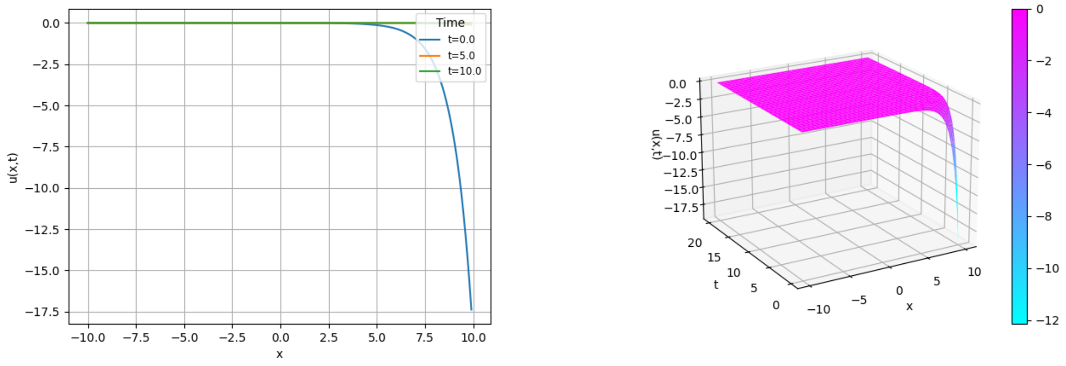
$$F' = c_1 e^{\tau s_2}, \tag{4.64}$$

now, substituting eqs. (4.63) and (4.64) into eq. (4.54) also considering that $u(x,t) = \mathcal{G}(s_2)$ where $s_2 = x - \alpha t$, the solution is,

$$u(x,t) = \frac{-1}{\kappa}\left(\frac{c_1 \tau e^{\tau(x-\alpha t)}}{c_1 e^{\tau(x-\alpha t)} + \tau c_2}\right), \tag{4.65}$$

where $c_i, (i = 1, 2)$ are the constants of integration.

Figure 4.4: Pressure profile in 2D and 3D for eq. (4.65). This is for $-10 \leq x \leq 10$, $0 \leq t \leq 20$, $c_i = 1$ and $\alpha = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

The sudden descent in acoustic pressure as witnessed in the above simulation renders this wave structure a kink solitary wave. The negative wave amplitude renders this wave structure as a dark solitary wave.

Case 2: $A_0 = \frac{-\tau}{\kappa}$, $A_1 = \frac{-1}{\kappa}$

Substituting the relevant arbitrary constants into eq. (4.58) yields the following ordinary differential equation

$$F'' + \tau F' = 0. \tag{4.66}$$

The exact solution of eq. (4.66) and its derivative are provided below respectively,

$$F = -\frac{c_1}{\tau} e^{-s_2 \tau} + c_2, \tag{4.67}$$

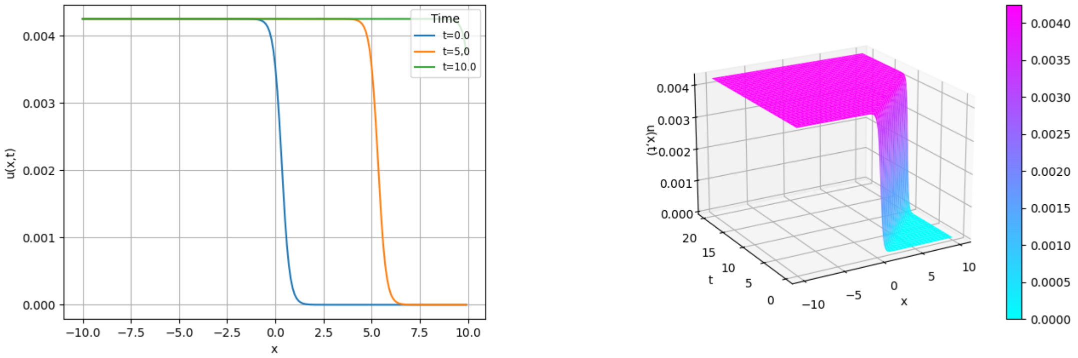
$$F' = c_1 e^{-s_2 \tau}. \tag{4.68}$$

Substituting eqs. (4.67) and (4.68) into eq. (4.54) gives the following exact solution

$$u(x,t) = \frac{-\tau}{\kappa} - \frac{1}{\kappa}\left(\frac{c_1 \tau}{\tau c_2 e^{\tau(x-\alpha t)} - c_1}\right), \tag{4.69}$$

such that $c_i, i = 1, 2$, are the constants of integration.

Figure 4.5: Pressure profile in 2D and 3D for eq. (4.69). This is for $-10 \leq x \leq 10$, $0 \leq t \leq 20$, $c_i = 1$ and $\alpha = 1$.



(a) 2D-Pressure graph

(b) 3D-Pressure graph

Figure 4.5 is a depiction of a kink wave solitary wave solution, which is evidenced by the lack of any temporal evolution in shape or size. The positive wave amplitude implies this wave structure is a bright solitary wave solution.

4.6 Conclusion

In this chapter, we performed Lie symmetry analysis for eq. (4.1), this model has the linear wave relation incorporated for the dissipation term. We found the associated Lie point symmetries which were used to construct one-parameter Lie groups and exact invariant solutions. Furthermore, the graphical representations of the exact solutions were provided. We observed several wave structures such as kinks, exponential, periodic, bright and dark. In Figure 4.3, we observed a hybrid waveform which is a mixture of periodic, kink and dark solitary wave structures.

Chapter 5

Conclusions

The propagation of acoustic waves in thermoviscous fluids is described using complicated, non-linear models such as the Westervelt model. As such, a breakthrough in finding any exact solution carries an immense amount of interest in the scientific community. Our work mainly focused on finding a family of closed-form, non-trivial solutions to the partial differential equations associated with the Westervelt model through the application of Lie symmetry analysis. This comprised computing Lie point symmetries, Lie point transformations, optimal systems of sub-algebras, symmetry reductions, invariant sub-models and group invariant solutions.

In Chapter 3, we focused on the model given by eq. (3.1) and in Chapter 4 we dealt with the second model in eq. (4.1). We generated an overdetermined system of linear homogeneous partial differential equations in ξ^1, ξ^2 and η using a symbolic computation package (YaLie). Upon solving system under consideration, we were able to obtain the infinitesimal operator which was used to construct Lie point symmetries. Furthermore, the corresponding optimal system of one-dimensional sub-algebras was constructed and used for symmetry reductions resulting in invariant sub-models. We then proceeded to determining the one-parameter Lie groups corresponding to each model and the closed form invariant solutions.

Some of the solitary wave solutions that admit the general travelling wave solutions were found by implementing the modified simple equation method. The analytical solutions obtained using this method in both models have been tested and were indeed found to be the solutions for our respective models[1].

For future research, as a way to obtain additional invariant solutions associated with the Westervelt model, one could utilise Lie symmetry analysis to tackle this problem under fractional formulation. This means introducing fractional derivatives in the time domain to account for memory effects and non-local interactions in the medium.

---

[1] See the Appendix 5.

Appendix

Example 1. The set,

$$T_{\epsilon_1}: \quad \bar{x} = xe^{3C\epsilon_1}, \quad \bar{t} = te^{2C\epsilon_1}, \quad \bar{u} = ue^{-2C\epsilon_1}, \quad \epsilon_1 \in \mathbb{R}^+, T_{\epsilon_1} \in G,$$

forms a one parameter Lie group.

Proof

The composition of the above transformations is given as follows,

$$\begin{aligned} \bar{\bar{x}} &= \bar{x}e^{3C\bar{\epsilon}_1}, \\ &= \bar{x}e^{3C\bar{\epsilon}_1}, \\ &= xe^{3C(\bar{\epsilon}_1+\epsilon_1)}, \end{aligned}$$

similarly $\bar{\bar{t}} = te^{2C(\bar{\epsilon}_1+\epsilon_1)}$ and $\bar{\bar{u}} = ue^{-2C(\bar{\epsilon}_1+\epsilon_1)}$. Thus, $\bar{\bar{x}} = xe^{3C\pi}$, $\bar{\bar{t}} = te^{2C\pi}$ and $\bar{\bar{u}} = ue^{-2C\pi}$, where $\pi = \varphi(\bar{\epsilon}_1, \epsilon_1) = \bar{\epsilon}_1 + \epsilon_1$ is the composition law of the group. The closure property is established since $T_{\epsilon_1}T_{\bar{\epsilon}_1} = T_\pi \in G$, where $\pi = \bar{\epsilon}_1 + \epsilon_1$. Since $\bar{x} = x$, $\bar{t} = t$ and $\bar{u} = u$ when $\epsilon_1 = 0$, then T_0 is the identity of the group. Also, $T_kT_{\epsilon_1} = T_{\epsilon_1}T_k = T_0$ if and only if $k = -\epsilon_1$ for all $T_{\epsilon_1} \in G$, then the inverse of the group is given by $T_{-\epsilon_1}$. Since the properties of closure, identity, inverse hold we conclude that the set T_{ϵ_1} forms a one parameter Lie group □

Below, is how we have implemented Mathematica to demonstrate that eq. (3.79) is a solution of Model I. A similar approach can be followed for the other solutions.

$\boldsymbol{\kappa = \frac{2C}{\alpha}}$

$\boldsymbol{\tau = \frac{B}{\alpha} - \frac{A}{\alpha^3}}$

$\boldsymbol{c_1 = 1}$

$\boldsymbol{c_2 = 1}$

$\boldsymbol{\alpha = 1}$

$\boldsymbol{u[x,t] = \frac{-2}{\kappa}\left(\frac{c_1\tau}{c_2\tau e^{\tau(x-\alpha t)} - c_1}\right)}$

pde $\boldsymbol{= D[u[x,t],t,t,t] + AD[u[x,t],x,x] - BD[u[x,t],t,t] + CD[u[x,t]u[x,t],t,t]}$

$2C$

$-A+B$

1

1

1

$$-\frac{-A+B}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)}$$

$$\frac{6(A-B)^3(-A+B)^4e^{3(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^4} - \frac{2(A-B)^3(-A+B)^3e^{2(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^3} + \frac{4(A-B)^2(-A+B)^4e^{2(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^3} + \frac{(A-B)^3(-A+B)^2e^{(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^2} + C\left(\frac{6(A-B)^2(-A+B)^4e^{2(-A+B)(-t+x)}}{C^2\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^4} - \frac{2(A-B)^2(-A+B)^3e^{(-A+B)(-t+x)}}{C^2\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^3}\right) - B\left(-\frac{2(A-B)^2(-A+B)^3e^{2(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^3} + \frac{(A-B)^2(-A+B)^2e^{(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^2}\right) + A\left(-\frac{2(-A+B)^5e^{2(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^3} + \frac{(-A+B)^4e^{(-A+B)(-t+x)}}{C\left(-1+(-A+B)e^{(-A+B)(-t+x)}\right)^2}\right)$$

Simplify[pde]

0

Bibliography

[1] AITKEN, M. J. George Bidel Airy and his contribution to optometry. *Clinical and Experimental Optometry 49*, 5 (1966), 131–133.

[2] ANCO, S. C., MÁRQUEZ, A. P., GARRIDO, T. M., AND GANDARIAS, M. L. Symmetry analysis and hidden variational structure of Westervelt's equation in nonlinear acoustics. *Communications in Nonlinear Science and Numerical Simulation 124* (2023), 107315.

[3] ANDERSON, R. D. One-dimensional continuous curves and a homogeneity theorem. *Annals of Mathematics 68*, 1 (1958), 1–16.

[4] ARRIGO, D. J. *Symmetry analysis of differential equations: an introduction*. John Wiley & Sons, 2015.

[5] ASLAM, M., ISLAM, N., ZAHEER, M. M., AND ALAM, M. Comparative response of double hinged alp using Airy's and Stokes'wave theories. *International Journal of Innovative Research in Science, Engineering & Technology 2*, 5 (2013), 1532–1539.

[6] BAKER, K., BANJAI, L., AND PTASHNYK, M. Numerical analysis of a time-stepping method for the Westervelt equation with time-fractional damping. *Mathematics of Computation 93* (2024), 2711–2743.

[7] BEYER, R. T. *Sounds of our times: Two hundred years of acoustics*. Springer Science & Business Media, 1999.

[8] BISWAS, A., YILDIRIM, Y., YASAR, E., ZHOU, Q., MOSHOKOA, S. P., AND BELIC, M. Optical solitons for Lakshmanan–Porsezian–Daniel model by modified simple equation method. *Optik 160* (2018), 24–32.

[9] BLACKSTOCK, D. T., CORMACK, J. M., AND HAMILTON, M. F. Early history of nonlinear acoustics. *Proceedings of Meetings on Acoustics 36*, 1 (2019), 045007.

[10] BLEAKNEY, W., AND TAUB, A. Interaction of shock waves. *Reviews of modern physics 21*, 4 (1949), 584.

[11] BLUMAN, G. W., AND COLE, J. D. *Similarity methods for differential equations*. Springer Science & Business Media, 2012.

[12] BRADING, K., AND BROWN, H. R. Symmetries and Noether's theorems. *Symmetries in physics: Philosophical reflections* (2003), 89–109.

[13] BUICK, J., BUCKLEY, C., GREATED, C., AND GILBERT, J. Lattice Boltzmann BGK simulation of nonlinear sound waves: The development of a shock front. *Journal of Physics A: Mathematical and General 33*, 21 (2000), 3917.

[14] BURGERS, J. A mathematical model illustrating the theory of turbulence. *Advance in Applied Mechanics 1* (1948), 171–199.

[15] CANTWELL, B. J. *Introduction to symmetry analysis*. Cambridge University Press, 2002.

[16] CANTWELL, B. J., AND MOULDEN, T. Introduction to symmetry analysis. *Applied Mechanics Reviews 57*, 1 (2004), B4.

[17] CHIRKUNOV, Y. A. Invariant solutions of the Westervelt model of nonlinear hydroacoustics without dissipation. *International Journal of Non-Linear Mechanics 85* (2016), 41–53.

[18] CHIRKUNOV, Y. A. Invariant submodels of the Westervelt model with dissipation. *International Journal of Non-Linear Mechanics 84* (2016), 139–144.

[19] COHEN, A. *An Introduction to the Lie Theory of one-parameter groups: With Applications to the solution of Differential Equations*. DC Health & Company, 1911.

[20] CRIGHTON, D. G. Model equations of nonlinear acoustics. *Annual Review of Fluid Mechanics 11*, 1 (1979), 11–33.

[21] Crighton, D. G., and Scott, J. F. Asymptotic solutions of model equations in nonlinear acoustics. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences 292*, 1389 (1979), 101–134.

[22] Curtis, C. W., Dzhamay, A., Hereman, W. A., and Prinari, B. *Nonlinear Wave Equations*. American Mathematical Soc., 2015.

[23] Doinikov, A. A., Novell, A., Calmon, P., and Bouakaz, A. Simulations and measurements of 3-D ultrasonic fields radiated by phased-array transducers using the Westervelt equation. *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control 61*, 9 (2014), 1470–1477.

[24] Dusunceli, F., Celik, E., Askin, M., and Bulut, H. New exact solutions for the doubly dispersive equation using the improved Bernoulli sub-equation function method. *Indian Journal of Physics 95* (2021), 309–314.

[25] El-Shiekh, R. M., Gaballah, M., and Elelamy, A. F. Similarity reductions and wave solutions for the 3D-Kudryashov–Sinelshchikov equation with variable-coefficients in gas bubbles for a liquid. *Results in Physics 40* (2022), 105782.

[26] Ellenbogen, T., Voloch-Bloch, N., Ganany-Padowicz, A., and Arie, A. Nonlinear generation and manipulation of Airy beams. *Nature photonics 3*, 7 (2009), 395–398.

[27] Enflo, B. O., and Hedberg, C. M. *Theory of Nonlinear Acoustics in Fluids*. Kluwer Academic Publishers, 2004.

[28] Euler, N. *Continuous symmetries, Lie algebras and differential equations*. PhD thesis, University of Johannesburg (South Africa), 1991.

[29] Fernando, R., Druon, Y., Coulouvrat, F., and Marchiano, R. Nonlinear waves and shocks in a rigid acoustical guide. *The Journal of the Acoustical Society of America 129*, 2 (2011), 604–615.

[30] FILIPPI, P. *Theoretical Acoustics and Numerical Techniques*. CISM International Centre for Mechanical Sciences. Springer Vienna, 2014.

[31] FORBES, L. K. The Rayleigh–Taylor instability for inviscid and viscous fluids. *Journal of Engineering Mathematics 65* (2009), 273–290.

[32] GARRETT, S. L. *Understanding acoustics: An experimentalist's view of sound and vibration*. Springer Nature, 2020.

[33] GAUTSCHI, W. Leonhard Euler: His life, the man, and his works. *SIAM review 50*, 1 (2008), 3–33.

[34] GHAZANFAR, S., AHMED, N., IQBAL, M. S., AKGÜL, A., BAYRAM, M., AND DE LA SEN, M. Imaging ultrasound propagation using the Westervelt equation by the generalized Kudryashov and modified Kudryashov methods. *Applied Sciences 12*, 22 (2022), 11813.

[35] GINSBERG, J. Multi-dimensional non-linear acoustic wave propagation, part I: An alternative to the method of characteristics. *Journal of Sound and Vibration 40*, 3 (1975), 351–358.

[36] GRIFFITHS, G. W., AND SCHIESSER, W. E. Linear and nonlinear waves. *Scholarpedia 4*, 7 (2009), 4308.

[37] GUPTA, P., AND SRIVASTAVA, A. Numerical analysis of thermal response of tissues subjected to high intensity focused ultrasound. *International Journal of Hyperthermia 35*, 1 (2018), 419–434.

[38] HADDADI, S., AND AHMADIAN, M. T. Analysis of nonlinear acoustic wave propagation in HIFU treatment using Westervelt equation. *Scientia Iranica 25*, 4 (2018), 2087–2097.

[39] HAMMERTON, P., AND CRIGHTON, D. G. Approximate solution methods for nonlinear acoustic propagation over long ranges. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences 426*, 1870 (1989), 125–152.

[40] HELGASON, S. Sophus Lie, the mathematician. In *Contenido en Proceedings of The Sophus Lie Memorial Conference, Oslo* (1994), pp. 3–21.

[41] HU, X., LI, Y., AND CHEN, Y. The construction of two-dimensional optimal systems for the invariant solutions. *arXiv: Group Theory* (2014). `https://api.semanticscholar.org/CorpusID:119587074`.

[42] HUMPHREY, V. F. Ultrasound and matter—physical interactions. *Progress in biophysics and molecular biology 93*, 1-3 (2007), 195–211.

[43] HUSSAIN, A., USMAN, M., AL-SINAN, B. R., OSMAN, W. M., AND IBRAHIM, T. F. Symmetry analysis and closed-form invariant solutions of the nonlinear wave equations in elasticity using optimal system of Lie subalgebra. *Chinese Journal of Physics 83* (2023), 1–13.

[44] IBRAGIMOV, N. H. *CRC handbook of Lie group analysis of differential equations*, vol. 3. CRC press, 1995.

[45] IBRAGIMOV, N. H., AND IBRAGIMOV, R. N. Internal gravity wave beams as invariant solutions of Boussinesq equations in geophysical fluid dynamics. *Communications in Nonlinear Science and Numerical Simulation 15*, 8 (2010), 1989–2002.

[46] IBRAGIMOV, N. K., AND IBRAGIMOV, N. K. *Elementary Lie group analysis and ordinary differential equations*. Wiley New York, 1999.

[47] JAWAD, A. J. M., PETKOVIĆ, M. D., AND BISWAS, A. Modified simple equation method for nonlinear evolution equations. *Applied Mathematics and Computation 217*, 2 (2010), 869–877.

[48] JORDAN, P. A survey of weakly-nonlinear acoustic models: 1910–2009. *Mechanics Research Communications 73* (2016), 127–139.

[49] KABIR, M., KHAJEH, A., ABDI AGHDAM, E., AND YOUSEFI KOMA, A. Modified Kudryashov method for finding exact solitary wave solutions of higher-order nonlinear equations. *Mathematical methods in the Applied Sciences 34*, 2 (2011), 213–219.

[50] KAIBE, B. C. *Application of Lie symmetries to Solving Partial Differential Equations associated with the Mathematics of Finance*. PhD thesis, University of Essex, 2021.

[51] KALTENBACHER, B., NIKOLIC, V., AND THALHAMMER, M. Efficient time integration methods based on operator splitting and application to the Westervelt equation. *IMA Journal of Numerical Analysis 35*, 3 (2015), 1092–1124.

[52] KALTENBACHER, B., AND RUNDELL, W. On the identification of the nonlinearity parameter in the Westervelt equation from boundary measurements. *Inverse Problems & Imaging 15* (2020), 1–27.

[53] KASTRUP, H. A. The contributions of Emmy Noether, Felix Klein and Sophus Lie to the modern concept of symmetries in physical systems. *Symmetries in physics (1600–1980)* (1987), 113–163.

[54] LEBLE, S. *Waveguide propagation of nonlinear waves.* Springer, 2019.

[55] LEBON, G. B., TZANAKIS, I., PERICLEOUS, K., ESKIN, D., AND GRANT, P. S. Ultrasonic liquid metal processing: The essential role of cavitation bubbles in controlling acoustic streaming. *Ultrasonics sonochemistry 55* (2019), 243–255.

[56] LURTON, X. *An introduction to underwater acoustics: Principles and Applications.* Springer Science & Business Media, 2002.

[57] MASATSUGU SEI SUZUKI, AND ITSUKO S. A particle confined within a triangular potential well: Airy function, with the use of Mathematica. `https://scholar.google.com/scholar?cluster=14508108011724008764&hl=en&as_sdt=0,5`, 2020.

[58] MAST, T. D. Empirical relationships between acoustic parameters in human soft tissues. *Acoustics Research Letters Online 1*, 2 (2000), 37–42.

[59] MAUGIN, G. A. *Continuum mechanics through the twentieth century*, vol. 10. Springer, 2013.

[60] MIRZAZADEH, M. Modified simple equation method and its applications to nonlinear partial differential equations. *Information Sciences Letters 3*, 1 (2014), 1.

[61] MIYAGIF, H., AND TANIGUCHI, T. Application of the Lagrange-Charpit method to analyse the power system's stability. *International Journal of Control 32*, 2 (1980), 371–379.

[62] MOLATI, M. *Group classification of coupled partial differential equations with applications to flow in a collapsible channel and diffusion processes*. PhD thesis, University of the Witwatersrand, 2010.

[63] MORENO, J. M. D. Lie symmetries. `https://library.wolfram.com/infocenter/MathSource/4231/`.

[64] NICHOLSON, W. K. *Introduction to abstract algebra*. John Wiley & Sons, 2012.

[65] NIKOLIC, V., AND WOHLMUTH, B. A priori error estimates for the finite element approximation of Westervelt's quasi-linear acoustic wave equation. *SIAM Journal on Numerical Analysis 57*, 4 (2019), 1897–1918.

[66] NORTON, G. V., AND PURRINGTON, R. D. The Westervelt equation with viscous attenuation versus a causal propagation operator: A numerical comparison. *Journal of Sound and Vibration 327*, 1-2 (2009), 163–172.

[67] NORTON, G. V., AND PURRINGTON, R. D. The Westervelt equation with a causal propagation operator coupled to the bioheat equation. *Evolution Equations & Control Theory 5*, 3 (2016), 449.

[68] O. VALLEE, M. S. *Airy functions and applications to physics*. Imperial College Press, 2010.

[69] OLVER, P. J. *Applications of Lie groups to differential equations*, vol. 107. Springer Science & Business Media, 1993.

[70] OVSYANNIKOV, L. V. *Lectures on the theory of group properties of differential equations*. World Scientific Publishing Company, 2013.

[71] PATERA, J., AND WINTERNITZ, P. Subalgebras of real three-and four-dimensional Lie algebras. *Journal of Mathematical Physics 18*, 7 (1977), 1449–1455.

[72] Pedersen, A. *Effects of nonlinear sound propagation in fisheries research.* PhD thesis, The University of Bergen, 2007.

[73] Pérez-García, V. M., Calvo, G. F., Belmonte-Beitia, J., Diego, D., and Pérez-Romasanta, L. Bright solitary waves in malignant gliomas. *Physical Review E 84*, 2 (2011), 021921.

[74] Qawaqneh, H., Zafar, A., Raheel, M., Zaagan, A. A., Zahran, E. H., Cevikel, A., and Bekir, A. New soliton solutions of M-fractional Westervelt model in ultrasound imaging via two analytical techniques. *Optical and Quantum Electronics 56*, 5 (2024), 737.

[75] Rayleigh, J. W. S. B. *The theory of sound*, vol. 2. Macmillan, 1896.

[76] Rossing, T. D. A brief history of acoustics. *Springer Handbook of Acoustics* (2014), 11–27.

[77] Rowe, D. E. The early geometrical works of Sophus Lie and Felix Klein. In *Ideas and their Reception*. Elsevier, 1989, pp. 208–273.

[78] Rudenko, O., and Soluian, S. The theoretical principles of nonlinear acoustics. *Moscow Izdatel Nauka* (1975).

[79] Salas, M. D. The curious events leading to the theory of shock waves. *Shock waves 16*, 6 (2007), 477–487.

[80] San, S., and Altunay, R. Application of the generalized Kudryashov method to various physical models. *Applied Mathematics I & Information Sciences Letters 8*, 2 (2021), 7–13.

[81] Sasaki, S. A characterization of contact transformations. *Tohoku Mathematical Journal, Second Series 16*, 3 (1964), 285–290.

[82] Seadawy, A. R., Younis, M., Baber, M. Z., Iqbal, M. S., and Rizvi, S. T. Nonlinear acoustic wave structures to the Zabolotskaya-Khokholov dynamical model. *Journal of Geometry and Physics 175* (2022), 104474.

[83] SEKHESA, T. *Nonlinear acoustics in medical ultrasound.* Bachelor's thesis, National University of Lesotho, 2023.

[84] SHAIKH, T. S., BABER, M. Z., AHMED, N., IQBAL, M. S., AKGÜL, A., AND EL DIN, S. M. Acoustic wave structures for the confirmable time-fractional Westervelt equation in ultrasound imaging. *Results in Physics 49* (2023), 106494.

[85] SOLOVCHUK, M., SHEU, T. W., AND THIRIET, M. Simulation of nonlinear Westervelt equation for the investigation of acoustic streaming and nonlinear propagation effects. *The Journal of the Acoustical Society of America 134*, 5 (2013), 3931–3942.

[86] STEEB, W.-H. *Continuous symmetries, Lie algebras, differential equations and computer algebra.* World Scientific Publishing Company, 2007.

[87] STRAUSS, W. A. *Nonlinear wave equations.* American Mathematical Soc., 1990.

[88] VANHILLE, C., AND CAMPOS-POZUELO, C. Acoustic cavitation mechanism: A nonlinear model. *Ultrasonics sonochemistry 19*, 2 (2012), 217–220.

[89] VELASCO-SEGURA, R., AND RENDÓN, P. L. A finite volume approach for the simulation of nonlinear dissipative acoustic wave propagation. *Wave Motion 58* (2015), 180–195.

[90] WESTERVELT, P. Parametric acoustic array. *Journal of the Acoustical Society of America 35* (1963), 535–537.

[91] WOLFRAM RESEARCH, I. Champain, IL, `https://www.wolfram.com/mathematica/new-in-12/`, 2019.

[92] YOKUS, A., DURUR, H., AHMAD, H., THOUNTHONG, P., AND ZHANG, Y.-F. Construction of exact traveling wave solutions of the Bogoyavlenskii equation by (G'/G, 1/G)-expansion and (1/G')-expansion techniques. *Results in Physics 19* (2020), 103409.

[93] YOUNAS, U., YAO, F., NASREEN, N., KHAN, A., AND ABDELJAWAD, T. On the dynamics of soliton solutions for the nonlinear fractional dynamical system: Application in ultrasound imaging. *Results in Physics* (2024), 107349.

[94] YULDASHEV, P. V., KARZOVA, M. M., KREIDER, W., ROSNITSKIY, P. B., SAPOZHNIKOV, O. A., AND KHOKHLOVA, V. A. HIFU beam: A simulator for predicting axially symmetric nonlinear acoustic fields generated by focused transducers in a layered medium. *IEEE transactions on ultrasonics, ferroelectrics, and frequency control 68*, 9 (2021), 2837–2852.

[95] ZABOLOTSKAYA, E., AND KHOKHLOV., R. Quasi-plane waves in the nonlinear acoustics of confined beams. *Soviet Physics Acoustics 8* (1969), 35–40.

[96] ZAYED, E. M. A note on the modified simple equation method applied to Sharma–Tasso–Olver equation. *Applied Mathematics and Computation 218*, 7 (2011), 3962–3964.

[97] ZHENG, X., CHEN, Y., AND ZHANG, H. Generalized extended tanh-function method and its application to (1+ 1)-dimensional dispersive long wave equation. *Physics Letters A 311*, 2-3 (2003), 145–157.

[98] ZHOU, G., CHEN, R., AND RU, G. Propagation of an Airy beam in a strongly nonlocal nonlinear media. *Laser Physics Letters 11*, 10 (2014), 105001.

[99] ZIJTA, M. *Solving The Westervelt Equation With Losses Using First And Second Order Finite Element Method.* Bachelor's thesis, Delft University of Technology, 2017.